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STABILITY OF RIGID BODY ASSEMBLAGES WITH DILATANT INTERFACIAL CONTACT SLIDING

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Abstract—The present paper considers a generalization of the classical theory of the stability of rigid body assemblages with dry joints based on the inclusion of slip-dependent dilatation in the contact mechanics of the interfaces. The friction angle φ is decomposed into two components, ρ and β, where ρ stands for the dissipative component and β stands for the purely nondissipative geometric component caused by the inclination of the asperities. The effect of each of these components on the overall stability is analysed separately and in combination. The investigation reveals the decisive significance of the nondissipative component β for the evaluation of the stability, not only when there are no dissipative effects, but also in combination with dissipative friction. If the boundaries of stability depend on contact sliding and the friction is purely dissipative, unique solutions are generally not attainable. Safe ranges can be established for plane structures. For three-dimensional structures, where $\rho > 0$, definite boundaries can be established in certain cases only if $\beta > 0$; otherwise the problem remains ambiguous.

NOMENCLATURE

$\mathbf{n}_\mu(r)$	outside normal of body (μ) at **r** on interface $\Gamma_{\nu\mu}$
$\mathbf{p}_{\mu\nu}(r)$	stress vector on $\Gamma_{\nu\mu}$ acting on body (μ)
$\mathbf{r}, \mathbf{r}_\nu, \mathbf{r}_{\nu\mu}, r$	position vectors
$\mathbf{s} = \{x,y\}^{\mathrm{T}}$	position vector in base plane $\pi^{\circ}_{\nu\mu}$
$\mathbf{t}$	Unit vector in $\pi^{\circ}_{\nu\mu}$
$\mathbf{T}$	tangent vector on $\Gamma_{\nu\mu}$
$\mathbf{u}_\nu$	displacement of body (ν)
$\mathbf{v}_\nu$	translation of body (ν)
$\boldsymbol{\omega}_\nu$	rotation of body (ν)
$[\mathbf{u}](r) = \mathbf{u}_\nu(r) - \mathbf{u}_\rho(r)$	discontinuity on $\Gamma_{\nu\rho}$
$\gamma_{\nu\mu}(r) = \mathbf{u}_\nu(r) - \mathbf{u}_\mu(r)$	deformation vector $\Gamma_{\nu\mu}$
$\mathbf{v}_{\nu\mu} = \mathbf{v}_\nu - \mathbf{v}_\mu$	
$\boldsymbol{\omega}_{\nu\mu} = \boldsymbol{\omega}_\nu - \boldsymbol{\omega}_\mu$	
$z(x, y)$	height of asperity
σ, σ_μ	normal stress in joint $\Gamma_{\nu\mu}$
τ, τ_μ	shear stress in joint $\Gamma_{\nu\mu}$
$w_{\nu\mu} = \{v_{\nu\mu}, \omega_{\nu\mu}\}_{\mathrm{T}}$	generalized deformation on $\Gamma_{\nu\mu}$
ϑ	angle of inclination of asperity
$\beta, \beta_\gamma = \max \vartheta_{\gamma'}$	angle of nondissipative friction
ρ	angle of dissipative friction
$\varphi = (\beta + \rho)_\tau$	angle of friction
$\mathbf{X}(\beta,r)$	cone of admissible $\gamma(r)$ at $\mathbf{r} \in \Gamma_{\nu\mu}$
$\Phi(\varphi,r)$	cone of friction at $\mathbf{r} \in \Gamma_{\nu\mu}$
$\mathbf{R},\mathbf{M}$	resultant force and moment
$S_{\mu\nu} = \{R_{\mu\nu}, M_{\mu\nu}\}^{\mathrm{T}}$	generalized forces on $\Gamma_{\nu\mu}$
P	load vector in space Z
$U = \{\mathbf{v}_\nu, \omega_\nu\}$	generalized load displacements
$E(\rho,\beta)$	cone of stability
$E_s(\rho,\beta)$	cone of safe loads
$E_n(\rho,\beta)$	cone of neutral loads
$E_a(\varphi)$	cone of admissible loads P^a
$K(\rho,\beta)$	cone of interface forces S
$\Lambda(\beta)$	cone of interface translations and rotations
$\Xi_a(\beta)$	cone of admissible displacements U^a.

1. INTRODUCTION

This paper considers the stability of an assemblage of rigid bodies interconnected by dry joints. Masonry structures belong to this category. The basic principles of their stability have actually been known intuitively from time immemorial. They were explicitly formulated as early as 300 years ago, but later repeatedly forgotten and rediscovered. The theory of masonry was brought to its present day basis by the work of Heyman (1966). This classical theory of stability was linked as a limit case to the theory of ideal plasticity and its consistent part was based on hinge mechanisms. In the following decades, the theory has developed along two main lines. One is directed towards an elastic continuum theory, where tensile traction and friction are totally eliminated (Del Piero, 1989; Como, 1992; Di Pasquale; 1992), and where the state of stress is reduced to compression fields. This applies especially to masonry made of small stones where the mortar has decayed. The other line is that of the classical theory dealing with discrete tension resistant ashlars. In this connection a remarkable contribution was recently made by Livesley (1992), who included friction with slip in the analysis of stability.

The main difficulty of the stability problem has been the absence of polarity with accompanying normality rules of the cone of friction $\mathbf{\Phi}$ and the cone $\mathbf{X}$ of displacement discontinuities γ in the joints $\Gamma_{\nu\mu}$ between bodies (ν) and (μ). This is reflected in the indefiniteness of the cone of stability E. In order to map out the possibilities of generalizing the classical theory to embrace explicitly also contact sliding, the present study investigates slip with dilatational effects. To this end, we use the friction law of Schneider (1976) for rough indented rock joints with the friction angle $\varphi = \rho + \vartheta$

$$|\tau| = -\sigma \tan(\rho+\alpha)\,;\sigma \leqslant 0; \quad \alpha = \vartheta \exp(-a(\sigma/\sigma_u))^b, \tag{1}$$

where ρ expresses the real dissipative friction and α depends on the inclination tgϑ of the asperity and the ultimate stress σ_u. If $\sigma/\sigma_u \ll 1$, then $\alpha \cong \vartheta$, and we get the simplified rule

$$|\tau| = -\sigma \tan(\rho+\vartheta)\,; \quad \sigma \leqslant 0. \tag{1'}$$

The addition $\varphi = \rho + \vartheta$ is to be applied vectorially because the angles ρ and ϑ are generally on different planes [Fig. 1(a)]. Here ρ is the actual friction angle or the greatest angle between the stress vector $\mathbf{p}_{\mu\nu}$ acting on a surface element $d\Gamma_{\mu\nu}$ of body μ, and the inside normal $-\mathbf{n}_\mu$ of $d\Gamma_{\mu\nu}$. ϑ is the angle between $-\mathbf{n}_\mu$ and $-\mathbf{k}$, the inside normal of a smoothed

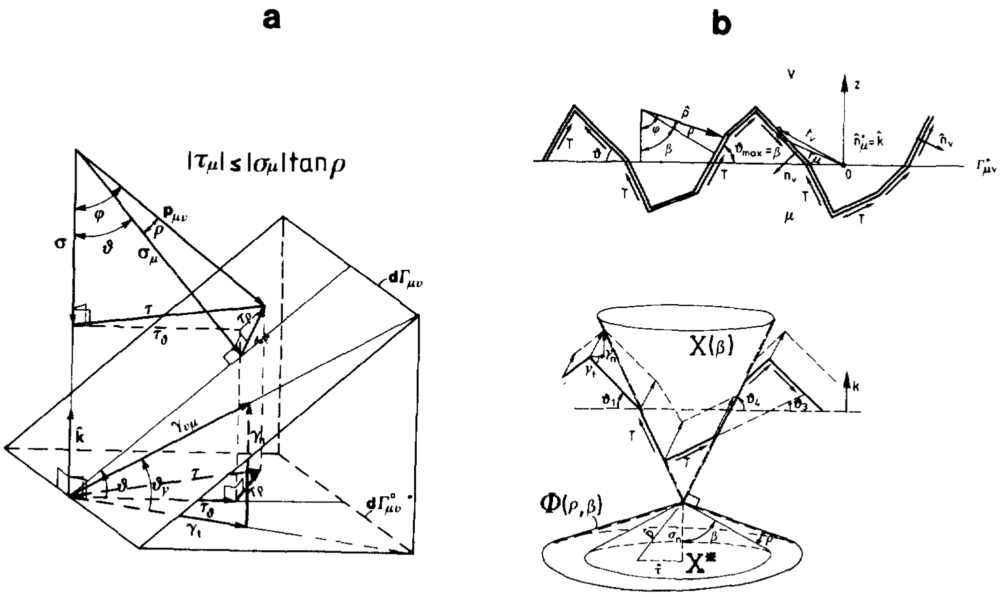

Fig. 1(a). Surface element $d\Gamma_{\nu\mu}$ of interface $\Gamma_{\mu\nu}$ with friction angles ρ, ϑ, $\varphi = \rho+\vartheta$; (b) cone $\mathbf{X}(\beta)$ of admissible deformations $\gamma = [\mathbf{u}]$ and cone of friction $\mathbf{\Phi}(\varphi)$.

middle surface $\Gamma^o_{\mu\nu}$ between the bodies μ and ν. Thus, ρ is associated with the dissipative component τ_ρ of the stress vector $\mathbf{p}$ in the tangent plane of the middle surface $\Gamma^o_{\mu\nu}$, and ϑ is associated with the nondissipative component τ_ϑ caused by the frictionless resistance to contact sliding along the inclined surface element $d\Gamma_{\mu\nu}$ [Fig. 1(b)].

This friction law of Schneider–Coulomb covers a wide variety of constitutive relations between joint tractions and deformations γ. It makes it possible to distinguish clearly between the dissipative component of the angle of friction ρ and the geometrical non-dissipative component ϑ, and to determine their effect, separately and in combination, on the overall stability of an assemblage of rigid bodies. The following analysis is, contrary to the classical theory, not based on plasticity but on the geometric impenetrability of the joints.

In order to restrict the extent, we often omit strict mathematical deductions by choosing a more heuristic approach. Because of the unilateral constraints, we cling exclusively to the concept of the positive cone with non-negative multiplicativity. In order to distinguish vectors in abstract spaces from those in the physical space R^3, we write only the latter with an extra bold letter. Thus $\mathbf{p}(r)$, $\gamma(r) \in R^3$, but $p(\cdot)$, $\gamma(\cdot) \in (L_2(\Gamma))^3$.

2. GEOMETRY OF THE DRY JOINTS

Let us consider a structure consisting of n members (ν) resting on a rigid foundation (o). Each member (ν), being a rigid body, occupies an open, bounded domain Ω_ν of R^3 with external boundary $\Gamma_{\nu e}$, and m_ν internal boundaries $\Gamma_{\nu\mu}$ that are dry joints to adjacent members (μ). The total number of internal boundaries of the assemblage is m. The interfaces $\Gamma_{\nu\mu}$ separating members (ν) and (μ) are assumed to be not necessary plane surfaces with complying approximately periodic corrugations, as in cracks. There is at least one Cartesian system (x, y, z) where $\Gamma_{\nu\mu}$:s position vectors $\mathbf{r}_\nu$, $\mathbf{r}_\mu$ may in the initial state be expressed using an appropriately chosen base plane $\pi^o_{\nu\mu}$

$$\mathbf{r}_{\nu\mu} = \mathbf{r}_\nu = \mathbf{r}_\mu = \mathbf{r}_{\nu\mu}(0) + \mathbf{s}(x, y) + z\,(x, y)\,\mathbf{k}\,; \quad \mathbf{s}(x, y) = x\mathbf{i} + y\mathbf{j} \in \Gamma^o_{\nu\mu}, \tag{2}$$

where $\mathbf{k}$ is the outside normal of member (μ) on $\pi^o_{\nu\mu}$, and $\Gamma^o_{\nu\mu}$ the projection of $\Gamma_{\nu\mu}$ on $\pi^o_{\nu\mu}$, $\mathbf{i}$, $\mathbf{j}$ are orthogonal unit vectors in $\Gamma^o_{\nu\mu}$ and $z\,(x, y)$ is a continuous piecewise smooth, univalued function (Fig. 1). The lengths l and amplitudes $\Delta z_{\max}$ of the periodical corrugations $z\,(x, y)$ are assumed to be small compared with the dimension L of $\Gamma^o_{\nu\mu}$

$$1 \ll L\,; \Delta z_{\max} \ll L. \tag{3}$$

The inclination $\tan\vartheta(s, t)$ of $z(s)$ in direction $\mathbf{t}$, where $\mathbf{t}$ is a unit vector in $\Gamma^o_{\nu\mu}$, is defined by $\nabla z = \operatorname{grad} z$, through the relation (Fig. 1)

$$\tan\vartheta\,(s, t) = \mathbf{t}\cdot\nabla z\,(s) > 0. \tag{4}$$

Because of the restraint imposed on $\mathbf{t}$, the directional tangent vector is

$$\mathbf{T}(s) = (\mathbf{t} + \tan\vartheta(s, t)\,\mathbf{k})\cos\vartheta\,(s, t). \tag{5}$$

The outside normal $\mathbf{n}_\nu(s)$ of a surface element $d\Gamma_{\mu\nu} \in \Gamma_{\mu\nu}$ of Ω_ν is a well defined, though multivalued function of s, which satisfies $\mathbf{k}\cdot\mathbf{n}_\nu(s) < 0$. The normal to a smooth surface element $d\Gamma_{\mu\nu}(s)$ is

$$\mathbf{n}_\mu(s) = -\mathbf{n}_\nu(s) = (\mathbf{r}_{,x} \times \mathbf{r}_{,y})/|\,\mathbf{r}_{,y} \times \mathbf{r}_{,x}| = (\mathbf{k} - \nabla z(s))/(1 + (\nabla z(s))^2)^{-1/2}. \tag{6}$$

Every directional tangent vector $\mathbf{T}(s)$ in the period corresponds to an orthogonal normal $\mathbf{n}_\mu(s)$

$$\mathbf{T}(s)\cdot\mathbf{n}_\mu(s) = 0. \tag{7}$$

3. DISPLACEMENTS AND DEFORMATIONS

The displacements $\mathbf{u}$ and rotations $\boldsymbol{\omega}$ of the structure are assumed to be small compared with the linear dimensions L of the members $|u| \ll L$, $|\omega| \ll 1$. Because of the rigidity of the bodies, the displacement field in each member (ν) is expressed by

$$\mathbf{u}_o = 0; \quad \mathbf{u}_\nu = \mathbf{v}_\nu + \boldsymbol{\omega}_\nu \times \mathbf{r}; \quad \mathbf{r}\in\Omega_\nu; \quad \nu = 1,\ldots,n. \tag{8}$$

This field forms a $6n$-dimensional subspace of a Hilbert space of displacements $(L_2)^3$ determined by the $6n$ generalized displacements $U_{k\nu}$, the $3n$ translations $v_{k\nu}$ and $3n$ rotations $\omega_{k\nu}$, $(k = x, y, z)$. The discontinuities $[\mathbf{u}]_{\nu\mu} = -[\mathbf{u}]_{\mu\nu} = \mathbf{u}_\nu(r) - \mathbf{u}_\mu(r)$ of the displacement field on the detachable surfaces $\Gamma_{\nu\mu}$ define the deformation vectors $\gamma_{\nu\mu}(r) = [\mathbf{u}]_{\nu\mu}$ of the joints

$$\boldsymbol{\gamma}_{\nu\mu}(r) = \gamma_n\mathbf{k} + \gamma_t\mathbf{t}; \quad \mathbf{r}\in\Gamma_{\nu\mu}. \tag{9}$$

where $\gamma_{\nu\mu} = -\gamma_{\mu\nu}$. The scalars γ_n and $|\gamma_t| = |\gamma_x\mathbf{i} + \gamma_y\mathbf{j}|$ characterize the dilatation and the sliding deformation, respectively, on $\Gamma_{\nu\mu}$. Because of eqn (8) and (9), and denoting $\mathbf{v}_{\nu\mu} = \mathbf{v}_\nu - \mathbf{v}_\mu$, $\boldsymbol{\omega}_{\nu\mu} = \boldsymbol{\omega}_\nu - \boldsymbol{\omega}_\mu$, there applies

$$\gamma_{\nu\mu}(r) = \mathbf{v}_{\nu\mu} + \boldsymbol{\omega}_{\nu\mu}\times\mathbf{r}; \quad \forall\mathbf{r}\in\Gamma_{\nu\mu}; \quad \nu = 1,\ldots,n; \quad \mu = 0,1,\ldots,n. \tag{8'}$$

Owing to the impenetrability of matter, the deformations γ on the interfaces are subjected to unilateral constraints.

The impenetrability condition

No point $\mathbf{r}_\mu$ of body (μ) can penetrate the interior Ω_ν of body (ν).

This can be replaced by the condition that any gap vector $\mathbf{g} = g\mathbf{k}$ in direction z, induced by $\mathbf{u}$ within the period $\Delta\Gamma_i$, satisfies $g \geqslant 0$. Denoting $\mathbf{r}'_\lambda(s) = \mathbf{r}_\lambda(s) + \mathbf{u}_\lambda(s)$; $z'_\lambda(s) = z_\lambda(s) + u_{z\lambda}(s)$, $(\lambda = \nu,\mu)$, and with $\Delta z = z(s+\Delta s) - z(s)$; $\mathbf{r}_\nu(s) = \mathbf{r}(s)$; $\mathbf{r}_\mu(s+\Delta s) = \mathbf{r}(s) + \Delta\mathbf{s} + \Delta z\mathbf{k}$; $\Delta\mathbf{s} = |\Delta s|\mathbf{t}$; $\mathbf{u}_\mu(s+\Delta s) = \mathbf{v}_\mu + \boldsymbol{\omega}\times(\mathbf{r}(s) + \Delta\mathbf{s} + \Delta z\mathbf{k})$, we obtain

$$\mathbf{r}'_\nu = \mathbf{r}(s) + \mathbf{u}_\nu(s); \quad \mathbf{r}'_\mu(s+\Delta s) = \mathbf{r}(s) + \mathbf{u}_\mu(s) + \Delta\mathbf{s} + \Delta z\mathbf{k} + \boldsymbol{\omega}_\mu\times(\Delta\mathbf{s} + \Delta z\mathbf{k}). \tag{8''}$$

According to eqn (8″), the gap vector is $\mathbf{g}(s,\Delta s) = \mathbf{r}'_\nu(s) - \mathbf{r}'_\mu(s+\Delta s) = g(s,\Delta s)\mathbf{k} + \mathbf{g}_t(s,\Delta s)$, where

$$g(s,\Delta s) = \gamma_n(s) - \Delta z - (\boldsymbol{\omega}_{\mu t}\times\Delta\mathbf{s})\cdot\mathbf{k}; \quad \mathbf{g}_t(s,\Delta s) = \gamma_t(s) - \Delta\mathbf{s} - \boldsymbol{\omega}_{\mu t}\times\mathbf{k}\Delta z. \tag{10}$$

Because of the smallness of γ, $\Delta\mathbf{s}$ and ω_μ, their products can be overlooked and the deformation $\gamma_{\nu\mu}(s)$ in each period $\Delta\Gamma_i$ can be considered constant γ_i, hence,

$$\mathbf{g}(s,\Delta s) = \gamma_t - \Delta\mathbf{s} + (\gamma_n - \Delta z)\mathbf{k}.$$

Since $\mathbf{g}\cdot\mathbf{t} = 0$, we get $\Delta\mathbf{s} = \gamma_t = |\gamma_t|\mathbf{t}$, and with $\Delta z = \nabla z\cdot\gamma_t = |\gamma_t|\tan\vartheta(s,t)$,

$$g(s,\Delta s) \cong \gamma_n - |\gamma_t|\tan\vartheta(s,t) \geqslant 0; \quad \forall s\in\Delta\Gamma_i. \tag{11}$$

The condition (11) for a given $\mathbf{t}$ must be valid for every $s\in\Delta\Gamma_i$. Hence, for any direction $\mathbf{t} = \gamma_t/|\gamma_t|$ and every $\mathbf{s}\in\Delta\Gamma_i$,

$$\gamma_n \geqslant |\gamma_t| \sup_{s \in \Delta\Gamma_i} (\tan \vartheta(s, t)). \tag{12}$$

Denoting $\sup(\tan \vartheta(s, t)) = \tan \beta_\gamma$, the impenetrability condition for any period $\Delta\Gamma_i$ can be written

$$\gamma_n \geqslant |\gamma_t| \tan \beta_\gamma; \quad \forall\, \Delta\Gamma_i \in \cup\, \Gamma_{\nu\mu}, \tag{12'}$$

where $\tan \beta_\gamma$ is the maximum inclination of the period $\Delta\Gamma_i$ in direction γ_t.

Lemma 1. The set of admissible vectors γ in a period $\Delta\Gamma_i$ of $\Gamma_{\nu\mu}$ is a closed convex cone $\mathbf{X}(\beta, r_i) \in R^3$, with the origin in $\mathbf{r}_i \in \Delta\Gamma_i$, characterized by the maximum ascent $\tan \beta_\gamma$ in direction γ_t.

Proof. $\mathbf{X}$ is a cone because of eqn (12) and $\gamma_n \geqslant 0$. The closeness follows from eqn (12′). Since eqn (11) can be written as

$$\gamma_n - \nabla z(s) \cdot \gamma_t \geqslant 0; \quad \mathbf{s} \in \Delta\Gamma_i,$$

any inequality for a fixed $\mathbf{s} \in \Delta\Gamma_i$ defines in R^3 a half-space of γ vectors bounded by a plane through the origin. The set of all the γ vectors that satisfy eqn (12′) for every $\mathbf{s} \in \Delta\Gamma$ is therefore the intersection of all these half-spaces, which is a convex set. Hence the cone $\mathbf{X}(\beta, r_i)$ is convex. □

The impenetrability condition (12′) restricts the set of admissible γ in each period $\Delta\Gamma_i$ to a cone $\mathbf{X}_i$ with the closed boundary $\partial\mathbf{X}$, whose generators are the directional tangents $\mathbf{T}_\gamma$ of steepest ascent in direction γ_t

$$\gamma = |\gamma_t| (\mathbf{t} + \mathbf{k} \tan \beta_\gamma) = |\gamma_t| \mathbf{T}_\gamma. \tag{13}$$

In this way, the impenetrability of an interface $\Gamma_{\mu\nu}$ with arbitrary continuous asperities, which satisfy eqns (3) and (4), can be expressed by a discrete set of convex cones. In the limit where $l \to 0$, with $\mathbf{X}(\beta, r_i)$ retaining its shape, the periods $\Delta\Gamma$ as well as the deformations are infinitesimal and $\gamma_{\nu\mu}(r) \in \mathbf{X}(\beta_\gamma, r)\ \forall\, \mathbf{r} \in \Gamma_{\nu\mu}$, where $\beta_\gamma = \beta(r, \mathbf{t}_\gamma)$ and $z(x, y)$ is a nonsmooth function.

4. STRESS TRANSFERENCE AND ITS INTERACTION WITH THE DEFORMATIONS IN DRY JOINTS

Denoting by $\mathbf{p}_{\mu\nu}(r)$ in $\Gamma_{\nu\mu}$ the stress vector acting on member (μ) and by $\mathbf{p}_{\nu\mu}(r)$, the vector acting on member (ν), where $\mathbf{r} \in \Gamma_{\nu\mu}$, we have the equilibrium condition

$$\mathbf{p}_{\mu\nu}(r) = -\mathbf{p}_{\nu\mu}(r). \tag{14}$$

In the local system defined by base vectors $\mathbf{n}_\mu(s)$ and $\mathbf{T}(s)$, the stress vector acting on an element $d\Gamma(s)$ of period $\Delta\Gamma_i \in \Gamma_{\mu\nu}$ is

$$\mathbf{p}_{\mu\nu}(s) = \sigma_\mu(s)\mathbf{n}_\mu(s) + \tau_\mu \mathbf{T}(s); \quad \sigma_\mu(s) \leqslant 0; \quad |\tau_\mu| \leqslant |\sigma_\mu| \tan \rho \tag{14'}$$

which expresses the friction law of Schneider–Coulomb [Fig. 1(a)]. In the x, y, z system, the friction law is according to eqn (1):

$$\mathbf{p}_{\mu\nu}(s) = \sigma(s)\mathbf{k} + \tau(s)\mathbf{t}; \quad \sigma(s) \leqslant 0; \quad |\tau(s)| \leqslant |\sigma(s)| \tan(\vartheta(s) + \rho(s))_\tau; \quad \tan \rho \cdot \tan \beta < 1. \tag{14''}$$

The inequality tan ρ tan $\beta < 1$ warrants the nonpositivity of σ. The resultant acting on a period $\Delta\Gamma_{i\nu}$ is

$$\Delta\mathbf{R}_{\mu\nu} = \int_{\Delta\Gamma_i} \mathbf{p}_{\mu\nu}\,d\Gamma.$$

The set of vectors $\mathbf{p}_{\mu\nu}$ and $\Delta\mathbf{R}_{\mu\nu}$ on $\Delta\Gamma_i$ is restricted by the friction law (14″) to a cone $\mathbf{\Phi}_i(\rho,\vartheta)$. The generatrices of the lateral surface $\partial\mathbf{\Phi}$ of $\mathbf{\Phi}_i(\varphi)$ define the vectors $\mathbf{p}$ and $\Delta\mathbf{R}$ which induce sliding contact along inclined surface element $d\Gamma\in\Delta\Gamma_i$. According to Lemma 1, sliding is confined to elements $d\Gamma$ with the steepest ascent tan β. Therefore in Schneider–Coulomb's friction law, the angles $\vartheta(s)$ are restricted to their maximum values β_γ [Fig. 1b]. Hence the mean stress $\mathbf{p}_{\mu\nu} = \Delta\mathbf{R}_{\mu\nu}/|\Delta\Gamma_i|$ on $\Delta\Gamma_i$ is restricted to the friction cone

$$\mathbf{\Phi}_i(\varphi) = \{\mathbf{p}_{\mu\nu} = \sigma\mathbf{k}+\boldsymbol{\tau}\,;\sigma\leqslant 0\,;|\tau|\leqslant|\sigma|\tan\varphi\,;\varphi = (\rho+\beta)_\tau\,;\tan\rho\tan\beta<1\}. \quad (15)$$

In the limit where $\Delta\Gamma\to 0$, the stress vectors $\mathbf{p}_{\mu\nu}(r)$ are restricted to a cone $\mathbf{\Phi}(\varphi,r)$ of the same shape in R^3 [Fig. 1(b)].

$$\mathbf{\Phi}(\varphi,r) = \{\mathbf{p}(r) = \sigma(r)\mathbf{k}+\tau(r)\mathbf{t}\,;\sigma\leqslant 0\,;|\tau|\leqslant|\sigma|\tan\varphi\,;\tan\varphi = \tan(\beta_\rho+\rho_\rho)\,;$$
$$\tan\rho\tan\beta<1\,;\forall\,\mathbf{r}\in\Gamma_{\mu\nu}\}, \quad (15')$$

where

$$\tan\rho_\rho = \tan\rho(\cos\beta_\rho/\cos\beta)\,;\quad \tan\beta = |\nabla z|\,;\quad \tan\beta_\rho = \tan\beta\cos(\tau_\rho,\nabla z). \quad (15'')$$

Occasionally, we use the notations $\mathbf{\Phi}_i(\rho,\beta)$ for $\mathbf{\Phi}_i(\varphi)$ and $\mathbf{\Phi}(\rho,\beta,r)$ for $\mathbf{\Phi}(\varphi,r)$. The inequality $|\tau|<|\sigma|\tan\varphi$ defines the interior $\mathbf{\Phi}^{\circ}$ of $\mathbf{\Phi}$, and the equality $|\tau| = |\sigma|\tan\varphi$ determines the lateral surface $\partial\mathbf{\Phi}$ of $\mathbf{\Phi}(\varphi)$.

Because dynamic contact requires geometric contact, we have the following *correspondence rules* concerning $\mathbf{p}_{\mu\nu}$ and $\gamma_{\nu\mu}\in\Delta\Gamma_i$:

(i) if $\mathbf{p} = 0$ and $\gamma = 0$, only geometric contact;
(ii) if nonzero $\gamma\in\mathbf{X}_i^{\circ}$: $\mathbf{p} = 0$, no kinematic and no dynamic contact;
(iii) if nonzero $\mathbf{p}\in\mathbf{\Phi}_i^{\circ}$: $\gamma = 0$, dynamic contact at all points of $\Delta\Gamma_i$ (complete contact);
(iv) a sufficient condition for the simultaneous occurrence of nonzero $\mathbf{p}_{\mu\nu}(s)$ and $\gamma_{\nu\mu}(s)$ is $\mathbf{p}_{\mu\nu}(s)\in\partial\mathbf{\Phi}_i$; $\gamma_{\nu\mu}(s)\in\partial\mathbf{X}_i$ and s is a point of steepest ascent in $\Delta\Gamma$ (partial contact). In this case, the contact vectors $\mathbf{p}(s)$ and $\gamma(s)$ constitute corresponding generatrices of $\partial\mathbf{\Phi}$ and $\partial\mathbf{X}$, respectively,

$$\mathbf{p}(s) = |\sigma|(\tan\varphi\mathbf{t}_\tau-\mathbf{k})\,;\quad \gamma = |\gamma_t|(\mathbf{t}_\gamma+\tan\beta_\gamma\mathbf{k}). \quad (16)$$

(v) Vectors $\mathbf{p}''\in\Phi^{\circ}$, $\gamma'\in\mathbf{X}^{\circ}$ and nonzero $\mathbf{p}''\in\partial\mathbf{\Phi}$, $\gamma'\in\mathbf{X}^{\circ}$, or $\mathbf{p}\in\mathbf{\Phi}^{\circ}$ and nonzero $\gamma\in\partial\mathbf{X}$, respectively, are not corresponding vectors.

Generally, if $\gamma'(r)\in\mathbf{X}(\beta,r)$ and $\mathbf{p}''(r)\in\mathbf{\Phi}(\rho,\beta,r)$ then

$$\mathbf{p}'' = |\sigma''|((\boldsymbol{\tau}''/|\sigma''|)-\mathbf{k})\,;\quad \gamma' = |\gamma_t'|((\gamma_t'/|\gamma_t|)+(\gamma_n'/|\gamma_t'|)\mathbf{k}). \quad (16')$$

Taking into consideration $|\tau''|/|\sigma''|\leqslant\tan\varphi$ and $\gamma_n'/|\gamma_t'|\geqslant\tan\beta_{\gamma'}$ we obtain

$$\mathbf{p}''\cdot\gamma' = |\sigma''||\gamma_t'|((\boldsymbol{\tau}''\cdot\gamma_t')/(|\sigma''||\gamma_t'|)-\gamma_n'/|\gamma_t|)\leqslant|\sigma''||\gamma_t'|(\tan\varphi_\gamma-\tan\beta_{\gamma'}), \quad (16'')$$

where $\tan\varphi_{\gamma'} = \tan\varphi\cos(\tau'',\gamma_t')$; $\tan\beta_{\gamma'} = \tan\beta\cos(\tau'',\gamma_t')$. By assuming infinite strength and rigidity of the structural members, the above results make a highly rationalized interpretation of Schneider–Coulomb's empirical friction law. In reality the maximum tanβ

of the inclination of Schneider's friction law does not represent an actual inclination of the corrugation, but the combined effect of splinters and the remaining inclination of the fractured asperities caused by contact sliding. This then induces the dilatation γ_n of crack-like interfaces in brittle materials.

(a) *Nondissipative friction*

In this case, $\rho = 0$ and $\varphi = \beta$ and the cone of friction is $\mathbf{\Phi}(0,\beta)$. Every $\mathbf{p}_{\mu\nu}(s)$, where $\mathbf{s} \in \Delta\Gamma_i$, has the direction $\mathbf{n}_\nu(s)$ with the friction law according to eqn (14′) and

$$\mathbf{p}_{\mu\nu} = \sigma_\mu \mathbf{n}_\mu(s)\,; \quad \sigma_\mu \leqslant 0. \tag{17}$$

Because of eqns (12′), (16″) and (17) for $\mathbf{p}'' \in \mathbf{\Phi}(0,\beta)$, $\gamma' \in \mathbf{X}(\beta)$ holds the work inequality

$$\mathbf{p}''_{\mu\nu}(s) \cdot \gamma'_\nu\mu \leqslant 0\,; \quad \forall\, \mathbf{s} \in \Delta\Gamma_i \tag{18}$$

with equality only for corresponding vectors

$$\mathbf{p}_{\mu\nu} \cdot \gamma_{\nu\mu} = 0. \tag{19}$$

If a set of vectors g in a space G constitutes a cone K, the negative dual cone K^* of K is the set of all vectors g' in the dual space G', defined by the dual pairing $\langle g, g' \rangle \leqslant 0$. Because $(R^3)' = R^3$, from relation (18) there follows:

Lemma 2. The friction cone $\mathbf{\Phi}_i(0,\beta)$ is the negative polar cone $\mathbf{X}_i^*(\beta)$ of the cone $\mathbf{X}_i(\beta)$ of admissible gap vectors $\gamma_{\nu\mu}$: $\mathbf{\Phi}_i(0,\beta) = \mathbf{X}_i^*(\beta)$.

Lemma 2 implies that the cone $\mathbf{\Phi}_i(0,\beta)$ is convex, because it is the negative polar cone of the convex cone $\mathbf{X}_i(\beta)$. Because of eqns (18) and (19), we have

$$\mathbf{p} \cdot (\gamma' - \gamma) \geqslant 0\,; \quad \gamma \cdot (\mathbf{p}'' - \mathbf{p}) \geqslant 0\,; \quad \forall\, \gamma' \in \mathbf{X}_i(\beta)\,; \quad \forall\, \mathbf{p}'' \in \mathbf{\Phi}_i(0,\beta). \tag{20}$$

Plane sections parallel to the base plane $\Gamma^\circ_{\nu\mu}$ of the cones $\mathbf{\Phi}(0,\beta)$ and $\mathbf{X}(\beta)$ are convex curves C_τ, C_γ characterizing the vectors $\boldsymbol{\tau}$ and γ_t, respectively. From eqns (18) and (19) there follows

$$\boldsymbol{\tau} \cdot (\gamma'_t/\gamma'_n - \gamma_t/\gamma_n) \leqslant 0\,; \quad (\boldsymbol{\tau}''/|\sigma''| - \boldsymbol{\tau}/|\sigma|) \cdot \gamma_t \leqslant 0. \tag{21}$$

Because the common origin of γ_t and $\boldsymbol{\tau}$ is within C_τ and C_γ relations (19) and (20), they express the non-negativity of the slip work of corresponding vectors $\boldsymbol{\tau}$ and γ_t

$$\boldsymbol{\tau} \cdot \gamma_t \geqslant 0, \tag{22}$$

and in the limit where $\gamma' \to \gamma$, $\boldsymbol{\tau}'' \to \boldsymbol{\tau}$, they express the *sectional normality rule* where $\partial\mathbf{X}$ and $\partial\mathbf{\Phi}$ are smooth: $\boldsymbol{\tau} \cdot \mathrm{d}\gamma_t = 0$; $\gamma_t \cdot \mathrm{d}\boldsymbol{\tau} = 0$, which is a consequence of the general normality rule of eqn (19).

If the friction is orthotropic, the generatrix of cone $\mathbf{\Phi}_i(0,\beta)$ is $\mathbf{p} = \tau_x\mathbf{i} + \tau_y\mathbf{j} + \sigma\mathbf{k}$, whose components in the local x, y, z system satisfy

$$\partial\Phi(0,\beta) = \left(\frac{\tau_x}{\tan\beta_x}\right)^2 + \left(\frac{\tau_y}{\tan\beta_y}\right)^2 - \sigma^2 = 0\,; \quad \sigma < 0. \tag{23a}$$

The corresponding generatrix of $\mathbf{X}(0,\beta)$ is, according to the normality rule,

$$\gamma = \lambda\left(\frac{\partial\Phi}{\partial\tau_x}\mathbf{i}+\frac{\partial\Phi}{\partial\tau_y}\mathbf{j}+\frac{\partial\Phi}{\partial\sigma}\mathbf{k}\right) = 2\lambda\left(\frac{\tau_x}{\tan\beta_x^2}\mathbf{i}+\frac{\tau_y}{\tan\beta_y^2}\mathbf{k}-\sigma\mathbf{k}\right);\quad \lambda>0 \tag{23b}$$

from which the expression of the cone $\mathbf{X}_i(\beta)$ is obtained

$$\mathbf{X}_i(0,\beta) = (\gamma_x\tan\beta_x)^2+(\gamma_y\tan\beta_y)^2-\gamma_n^2 \leqslant 0;\quad \gamma_n>0. \tag{24}$$

This corresponds on $\Gamma_{\nu\mu}$ to a local fictitious asperity cone with a lateral surface $z(x',y')$ with inclination $\tan\beta_\gamma$

$$z = ((x'\tan\beta_x)^2+(y'\tan\beta_y)^2)^{1/2};\quad \tan\beta_\gamma = z/|\gamma_t| = ((\gamma_x\tan\beta_x)^2+(\gamma_y\tan\beta_y)^2)^{1/2}/(\gamma_x^2+\gamma_y^2)^{1/2}. \tag{25}$$

Conversely, from the normal of $\partial\mathbf{X}_i$, the expression (23a) of the friction cone $\mathbf{\Phi}_i(0,\beta)$ can be obtained. Although the normality rule (23b) formally conforms to the theory of plasticity, the physical background of nondissipative friction is quite different. This friction is based only on the impenetrability condition. The expression $\mathbf{\Phi}_i(0,\beta)$ actually represents the frictional potential, and $\mathbf{X}_i(\beta)$ represents the impenetrability potential.

(b) *Combined friction*

If the friction comprises a dissipative component ρ, this enlarges the friction cone. The cone of friction $\mathbf{\Phi}_i(\rho,\beta)$ is defined by the relations (15) or (15′). Since the dissipative friction ρ, because of plasticity, induces convex sets and $\mathbf{\Phi}(0,\beta,r)$ is convex, their sum is also a convex set.

Lemma 3. The set of admissible stress vectors $\mathbf{p}_{\mu\nu}(r)$ at point $\mathbf{r}$ of $\Gamma_{\nu\mu}$ is a closed convex cone $\mathbf{\Phi}(\rho,\beta,r)$.

This implies that the relations (20) and (21) for the curves C_γ and C_τ retain their validity unaltered. The scalar product $\mathbf{p}\cdot\gamma$ of corresponding nonzero vectors $\mathbf{p}$, γ defines the dissipative work $\mathrm{d}F(\sigma,\gamma,\varphi)$. $\mathrm{d}F$ expressed by base vectors $\mathbf{t}$, $\mathbf{k}$, according to eqn (16″), is

$$\mathrm{d}F = \mathbf{p}\cdot\gamma\,\mathrm{d}\Gamma = |\sigma||\gamma_t|(\tan\varphi_\gamma-\tan\beta_\gamma)\,\mathrm{d}\Gamma \geqslant 0. \tag{26}$$

The same expressed by base vectors $\mathbf{T}(s)$, $\mathbf{n}_\mu(s)$, according to eqns (13) and (14′), is

$$\mathrm{d}F = |\sigma_\mu||\gamma|\tan\rho\cdot\cos(\gamma,\tau_\mu)\,\mathrm{d}\Gamma \geqslant 0. \tag{26′}$$

If the friction is isotropic, the curves C_γ, C_τ are circles because $\cos(\tau,\gamma) = 1$. Then

$$\mathrm{d}F = \mathbf{p}\cdot\gamma\,\mathrm{d}\Gamma = |\sigma||\gamma_t|(\tan\varphi-\tan\beta)\,\mathrm{d}\Gamma;\quad \varphi = \beta+\rho. \tag{27}$$

5. THE POSSIBLE STATES OF EQUILIBRIUM AND DISPLACEMENT

In order to define the states of possible equilibrium (PE) and possible kinematics (PK), the following spaces and their duals are introduced (Fig. 2) by arranging indexes $\nu\mu = 1,\ldots,m$.

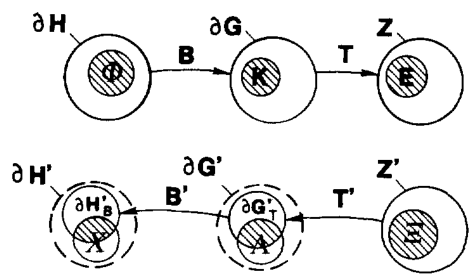

Fig. 2. Scheme of linear spaces, operators and their duals.

5.1. *Linear spaces and their duals*

5.1.1. $\partial H = L_2(\Gamma_c)^3$ space of stress $p_{\mu\nu}$ on interfaces $\cup\Gamma_{\mu\nu}$ including $p_{\mu 0}$ *on* $\Gamma_{\mu 0}$.

5.1.2. $\partial G = R^{6m}$ space of resultant forces $S_{\mu\nu}$ on $\cup\Gamma_{\mu\nu}$; $S_{\mu\nu} = \{\mathbf{R}_{\mu\nu}, \mathbf{M}_{\mu\nu}\}^{\mathrm{T}}$.

5.1.3. $Z = R^{6n}$ space of external loads $P_\nu = \{\mathbf{R}_\nu, \mathbf{M}_\nu\}^{\mathrm{T}}$ of assemblage $\cup\Omega_\nu$ for $\nu = 1,\ldots,n$, excluding $S_{\nu 0} \in \cup\Gamma_{\nu 0}$.

5.1.1′. $\partial H' = L_2(\Gamma_c)^3$ space of gap deformation $\gamma_{\nu\mu}$ on interfaces $\cup\Gamma_{\nu\mu}$ including $\gamma_{\mu 0}$ on $\cup\Gamma_{0\mu}$.

5.1.2′. $\partial G' = R^{6m}$ space of generalized deformations $w_{\nu\mu} = \{\mathbf{v}_{\nu\mu}, \boldsymbol{\omega}_{\nu\mu}\}^{\mathrm{T}}$ on $\cup\Gamma_{\nu\mu}$.

5.1.3′. $Z' = R^{6n}$ space of generalized displacements $U_\nu = \{\mathbf{v}_\nu, \boldsymbol{\omega}_\nu\}^{\mathrm{T}}$ of assemblage $\cup\Omega_\nu$.

5.2. *Linear operators and their duals*

5.2.1. $B:\partial H \rightarrow \partial G$; $B\mathrm{p}_{\mu\nu} = S_{\mu\nu}$; $\mu\nu = 1,\ldots,m$, where $B_{\nu\mu}\mathrm{p}_{\mu\nu} = \{\int 1\cdot\mathbf{p}_{\mu\nu}\mathrm{d}\Gamma_{\nu\mu}, \int\mathbf{r}\times\mathbf{p}_{\mu\nu}\mathrm{d}\Gamma_{\nu\mu}\}^{\mathrm{T}}$.

5.2.2. $T:\partial G \rightarrow Z$; $T_\nu S_{\mu\nu} = \Sigma S_{\mu\nu}$.

5.2.1′. $B':\partial G' \rightarrow \partial H'$; $B'w = \gamma_{\nu\mu}$ where $B_{\nu\mu}{}'w = \{1, \mathbf{r}_{\nu\mu}\mathrm{x}\}\cdot\{\mathbf{v}_{\nu\mu}, \boldsymbol{\omega}_{\nu\mu}\}^{\mathrm{T}}$.

5.2.2′. $T':Z' \rightarrow \partial G'$; $T'_\mu U_\nu = U_\nu - U_\mu$ if $\exists\,\Gamma_{\nu\mu}$.

Here $\mu = 1,\ldots,m_\nu$, $T'_\mu U_\nu = 0$ if $\nexists\,\Gamma_{\nu\mu}$ and B, B' and T, T' are diagonal matrix operators with m elements $B_{\nu\mu}$, $B_{\nu\mu}{}'$ and n elements T_μ, T'_μ, respectively.

5.3. *Equilibrium conditions*

5.3.1. $\mathbf{p}_{\mu\nu}(r) = -\mathbf{p}_{\nu\mu}(r)$ on $\forall\,\mathbf{r}\in\Gamma_{\nu\mu}$.

5.3.2. $\int\mathbf{p}_{\mu\nu}\,\mathrm{d}\Gamma_{\nu\mu} = \mathbf{R}_{\mu\nu}$; $\int\mathbf{r}\times\mathbf{p}_{\mu\nu}\mathrm{d}\Gamma_{\nu\mu} = \mathbf{M}_{\mu\nu}$.

5.3.3. $\Sigma\mathbf{R}_{\mu\nu} = \mathbf{R}_\nu$; $\Sigma\mathbf{M}_{\mu\nu} = \mathbf{M}_\nu$; $\mu = 1,\ldots,m_\nu$.

Kinematic conditions

5.3.1′. $\gamma_{\nu\mu} = -\gamma_{\mu\nu}$; $\mathbf{v}_{\nu\mu} = -\mathbf{v}_{\mu\nu}$; $\boldsymbol{\omega}_{\nu\mu} = -\boldsymbol{\omega}_{\mu\nu}$.

5.3.2′. $\mathbf{v}_{\nu\mu} + \boldsymbol{\omega}_{\nu\mu}\times\mathbf{r} = \gamma_{\nu\mu}$; $\forall\,\mathbf{r}\in\Gamma_{\nu\mu}$.

5.3.3′. $\{U_\nu - U_\mu\} = \{\mathbf{v}_{\nu\mu}, \boldsymbol{\omega}_{\nu\mu}\}^{\mathrm{T}}$; $\forall\,\nu\mu = 1,\ldots,m$.

Since the deformations $\gamma(\cdot)$, have the values $\gamma_{\nu\mu} = \mathbf{v}_{\nu\mu} + \boldsymbol{\omega}_{\nu\mu}\times\mathbf{r}$ in R^3, γ is a continuous function of r on every $\Gamma_{\nu\mu}$ in $\partial H'$, and there applies, according to 5.2.1′ and 5.2.2′,

$$\gamma(\cdot) = (B'w)(\cdot) = (B'T'U)(\cdot) \in \partial H'_B, \tag{28}$$

where $B'T':Z' \rightarrow \partial H'_B$ and $\partial H'_B$ is a $6n$-dimensional subspace of $\partial H'$. The set of states of possible equilibrium $\{p, S, P\}$ of the stress and force vectors $p\in\partial H$, $S\in\partial G$ and $P\in Z$ is determined by the equilibrium relations 5.2.1, 5.2.2, 5.3.1, 5.3.2, 5.3.3,

$$Bp = S; \quad BTp = TS = P. \tag{29}$$

The mappings $B:\partial H \rightarrow \partial G$ and $T:\partial G \rightarrow Z$ are subjective ($m > n$).

Work equations. According to relations 5.1, 5.2 and 5.3, the dual pairing of any possible state of equilibrium p'', S'', P'' and any possible state γ', w', U' satisfies the equations

$$\langle p'', \gamma'\rangle_{\partial H} = \langle p'', B'w'\rangle = \langle Bp'', w'\rangle = \langle S'', w'\rangle_{\partial G} = \langle S'', T'U'\rangle = \langle TS'', U'\rangle = \langle P'', U'\rangle_Z,$$

where $\mathbf{R}, \mathbf{M}\in R^3$; $\mathbf{v}, \boldsymbol{\omega}\in R_3$ in 5.1, 5.2 and 5.3 have been replaced by $S\in\partial G$, $P\in Z$, $w\in\partial G'$, $U\in Z'$.

The equations are, in reality, work equations concerning the internal work

$$\langle p'', \gamma'\rangle_{\partial H} = \Sigma\int\mathbf{p}_{\mu\nu}\cdot\gamma_{\nu\mu}\,\mathrm{d}\Gamma = \Sigma(\mathbf{R}_{\mu r}\cdot\mathbf{v}_{\nu\mu} + \mathbf{M}_{\mu\nu}\cdot\boldsymbol{\omega}_{\nu\mu}) = \langle S'', w'\rangle_{\partial G} \tag{30}$$

and the external work,

$$\langle P'', U'\rangle_Z = \Sigma(\mathbf{R}_\nu\cdot\mathbf{v}_\nu + \mathbf{M}_\nu\cdot\boldsymbol{\omega}_\nu),$$

respectively. The equations

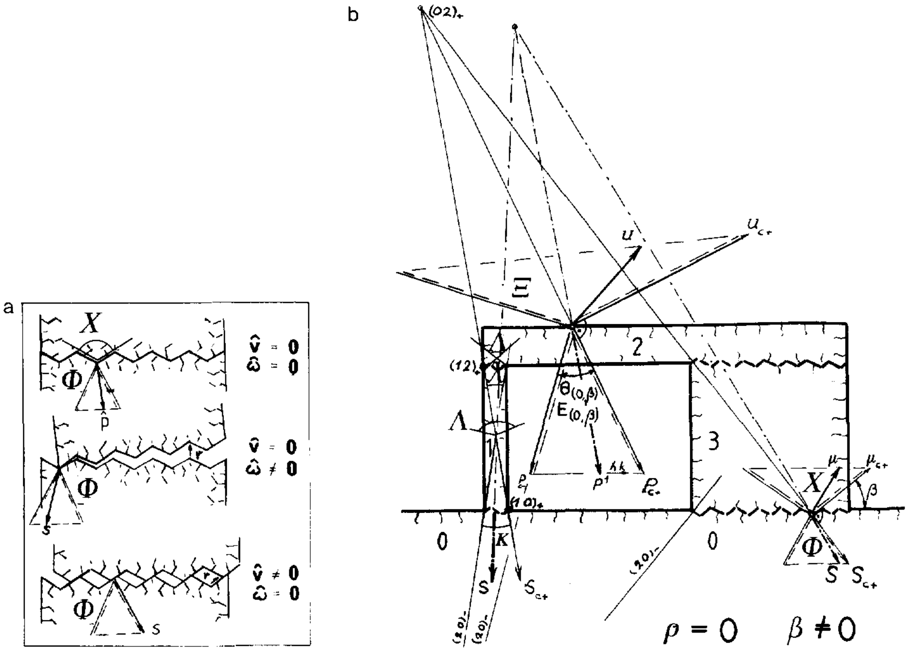

Fig. 3. Nondissipative friction $\varphi = \beta$: (a) cones $\mathbf{X}_a(\beta)$ and $\mathbf{\Phi}_a(\beta)$ of admissible deformations γ and admissible tractions p ;(b) the polarity of cone pairs (E_a, Ξ_a), (K_a, Λ_a), (X_a, Φ_a) is preserved on both global and local levels.

$$\langle p'', \gamma' \rangle_{\partial H} = \langle P'', U' \rangle_Z ; \quad \langle p'', \gamma' \rangle_{\partial H\nu\mu} = \langle S'', w' \rangle_{\partial G\nu\mu} \tag{31}$$

express the equality of the work of the load with the internal work in the joints and the equality of the internal work on $\Gamma_{\nu\mu}$ with the work of the resultant joint forces $S''_{\mu\nu}$, respectively.

6. ADMISSIBLE STATES AND STABILITY CRITERIA

The largest possible sets of admissible equilibrium (AE) and admissible kinematic states (AK) can be determined from Lemmas 1 and 3, respectively.

(a) *The set of admissible kinematic states* $AK = \{\gamma^a, w^a, U^a\}^T$

Proposition 1. The sets of admissible displacements $\{\gamma^a\}$, $\{w^a\}$, $\{U^a\}$, corresponding to angle β, constitute convex cones $X_a(\beta, \cdot)$, $\Lambda_a(\beta)$, $\Xi_a(\beta)$ in their respective spaces $\partial H'$, $\partial G'$, Z', $\Lambda_a(,126)$ $\Xi_a(\beta)$ being closed. There is a one to one correspondence between generatrix vectors $\gamma^c \in \partial X_a(\beta, \cdot)$, $w^c \in \partial\Lambda_a(\beta)$ and $U^c \in \partial\Xi_a(\beta)$, respectively.

Proof. In addition to the kinematic condition of sections 5.3.1′, 5.3.2′ and 5.3.3′, the value $\gamma(r)$ of $\gamma(\cdot)$ at every point $\mathbf{r} \in \cup\, \Gamma_{\nu\mu}$ is restricted to the local cone $\mathbf{X}(\beta, r)$ with the vertex in $\mathbf{r} \in \Gamma_{\nu\mu}$. The set $X(\beta, \cdot)$ of the functions subjected to restrictions defined by $\beta(r, \gamma_t)$ is

$$X(\beta, \cdot) = \{\gamma \in \partial H' \,|\, \gamma(r) \in \mathbf{X}(\beta, r), \beta \geqslant 0, \forall \mathbf{r} \in \cup\, \Gamma_{\nu\mu}\}. \tag{32}$$

Since every local $\mathbf{X}(\beta, r)$ is a closed convex cone, then if $\gamma(\cdot) \in X(\beta, \cdot)$, with $\lambda > 0$, also $\lambda\gamma(\cdot) \in X(\beta, \cdot)$ but $-\lambda\gamma(\cdot) \notin X(\beta, \cdot)$. If γ', $\gamma'' \in X(\beta, \cdot)$, which implies that every $\gamma'(r)$, $\gamma''(r) \in \mathbf{X}(\beta, r)$, then $(\gamma' + \gamma'')(\cdot) \in X(\beta, \cdot)$ because every $\mathbf{X}(\beta, r)$ is convex. Hence $X(\beta, \cdot)$ is a convex cone. Since also $\partial H'_B$ is convex, their intersection $X_a(\beta, \cdot) = X(\beta, \cdot) \cap \partial H'_B$, the set of admissible γ^a is a convex cone in $\partial H'$. Because the linear transformations B' and T' are injective, there is a one to one relation between γ^a and w^a and between γ^a and U^a, respectively (Fig. 2). Hence, to the convex cone $X_a \in \partial H'_B$, there corresponds a definite convex cone Λ_a

of vectors $w^a \in \partial G_T'$ and a definite convex cone $\Xi_a(\beta)$ of vectors $U^a \in Z'$, where $\partial G_T' = \{w = T'U;\ U \in Z'\} \subset \partial G'$ and $\partial H_B' = \{\gamma^a = B'w^a,\ w^a \in \partial G_T' \subset \partial G'\}$.

If $U_1 \in \Xi_a^\circ(\beta)$ and $q \notin \Xi_a(\beta)$, then $U(\gamma^c) = U_1 + \lambda^c q$ is a generatrix of $\partial \Xi_a(\beta)$, if $U(\lambda^c) \in \Xi(\beta)$, and $U(\lambda) \in \Xi_a(\beta)$, if $\lambda^c > \lambda > 0$. However, $U(\lambda)\, \Xi(\beta)$, if $\lambda > \lambda^c > 0$, because of the convexity and extent of Ξ_a (Luenberger, 1968). Therefore $\Xi_a(\beta)$ is closed. To any generatrix vector $\gamma^c \in \partial X_a(\beta, \cdot)$ there corresponds a unique generatrix vector w^c in $\partial \Lambda_a(\beta, \cdot)$ and a unique generatrix vector w^c in $\partial \Lambda_a(\beta, \cdot)$ and a unique generatrix vector U^c in $\partial \Xi_a(\beta, \cdot)$. □

The functions $\gamma^d(\cdot)$, whose values $\gamma(r) \in \mathbf{X}^\circ(\beta, r)$, represent fields where no geometric contact prevails on any interface $\Gamma_{\nu\mu}$. The set of all $\gamma^d(\cdot)$ generates the cone of *detachment* $X_d(\beta, \cdot) \subset X_a(\beta, \cdot)$. Corresponding cones of detachment $\Lambda_d(\beta) \subset \Lambda_a(\beta)$ and $\Xi_d(\beta) \subset \Xi_a(\beta)$ are generated by w^d and U^d, respectively. The *mechanisms* defined by the admissible fields $\gamma^b(\cdot) \notin X_d(\beta, \cdot)$, $w^b \notin \Lambda_d(\beta)$ and $U^b \notin \Xi_d(\beta)$, generate the cones of mechanisms $X_b(\beta, \cdot)$, $\Lambda_b(\beta)$ and $\Xi_b(\beta)$, respectively.

Corollary 1. (i) The cones of detachment $X_d(\beta, \cdot)$, $\Lambda_d(\beta)$, $\Xi_d(\beta)$ are open convex cones not containing the origin. (ii) The cones of mechanisms $\Lambda_b(\beta)$, $\Xi_b(\beta)$ are closed and contain the origin. This follows from Proposition 1 and

$$X_a(\beta, \cdot) = X_d(\beta, \cdot) \cup X_b(\beta, \cdot)\,; \quad \Lambda_a(\beta) = \Lambda_b(\beta) \cup \Lambda_b(\beta)\,; \quad \Xi_a(\beta) = \Xi_d(\beta) \cup \Xi_b(\beta). \quad (32')$$

(b) *The set of admissible equilibrium states AE*

In addition, to conditions 5.3.1, 5.3.2 and 5.3.3 there are the constraints imposed on $\mathbf{p}$ by the friction cones $\mathbf{\Phi}(\varphi, r)$, according to eqn (15). The set of functions $p(\cdot)$ restricted to these constraints is

$$\Phi(\varphi, \cdot) = \{p \in \partial H \,|\, \mathbf{p}(r) \in \mathbf{\Phi}(\varphi, r)\,, \forall \mathbf{r} \in \cup\, \Gamma_{\nu\mu}\}. \quad (33)$$

This set is a cone since if $p(\cdot) \in \Phi(\varphi, \cdot)$, then $\lambda \mathbf{p}(r) \in \mathbf{\Phi}(r)$ but $-\lambda \mathbf{p}(r) \notin \mathbf{\Phi}(r)$ if $\lambda > 0$. The set is convex because if $\mathbf{p}'(r) \in \mathbf{\Phi}(r)$ and $\mathbf{p}''(r) \in \mathbf{\Phi}(r)$, then also $\mathbf{p}'(r) + \mathbf{p}''(r) \in \Phi(r)$ because every $\mathbf{\Phi}(r)$ is convex. Hence $(p' + p'')(\cdot) \in \Phi(\varphi, \cdot)$ and $\Phi(\varphi, \cdot)$ is a convex cone.

The set Φ_a of admissible interface stresses p^a is

$$\Phi_a(\varphi, \cdot) = \{p \in \partial H \,|\, p \in (TB)^{-1} P \cap \Phi(\varphi, \cdot), P \in Z\,; \varphi > 0\}. \quad (34)$$

Actually, since any set $(TB)^{-1}P$, with P fixed, is convex, $(TB)^{-1}P \cap \Phi$ is also convex, and hence Φ_a is a convex cone. From the linearity of B and T there follows

$$Bp^a = S^a\,; \quad BTp^a = P^a\,; \quad TS^a = P^a. \quad (35)$$

Therefore the sets K_a and E_a of admissible S^a and P^a are convex cones.

The set $\Phi_m(\cdot) \subset \Phi_a(\cdot)$ defined by functions $p(\cdot)$, the values $\mathbf{p}(r)$ of which are restricted everywhere to the interior $\Phi^\circ(\varphi, r)$ of cone $\Phi(\varphi, r)$, characterizes *maximum dynamic contact*

$$\Phi_m(\varphi, \cdot) = \{p \in \partial H \,|\, \mathbf{p}(r) \in \mathbf{\Phi}_a^\circ(\varphi, r)\,; \forall\, \mathbf{r} \in \cup \Gamma_{\nu\mu}\}. \quad (36)$$

This set is convex and open and open convex cones K_m and E_m correspond to it. Proceeding as in Section 6.1, choosing $P_1 \in E_m(\varphi)$ and $k \notin E_a(\varphi)$, we can, because of the convexity and extent of $E_a(\varphi)$, define the generatrices $P^c = P + \lambda^c k$ of $\partial E_a(\varphi)$. Therefore the set $E_a(\varphi)$ is a closed cone. Denoting the complements of Φ_m, K_m, E_m within Φ_a, K_a, E_a, by Φ_n, K_n, E_n, respectively, we obtain:

Proposition 2. (i) The sets of admissible equilibrium $\{p^a\}$, $\{S^a\}$, $\{P^a\}$ constitute convex cones $\Phi_a(\varphi, \cdot)$, $K_a(\varphi)$ and $E_a(\varphi)$ in their respective spaces ∂H, ∂G and Z, the cones $K_a(\varphi)$, $E_a(\varphi)$ being closed. (ii) The cones of maximum dynamic contact $\Phi_m(\varphi, \cdot)$, $K_m(\varphi)$, $E_m(\varphi)$, are open convex cones not containing the origin. (iii) The sets $\Phi_n(\varphi)$, $K_n(\varphi)$, $E_n(\varphi)$ are closed cones containing the origin.

(c) *Stability criteria*

The states of actual equilibrium $\{p, S, P\}$ are further restricted by the correspondence rules (i)–(v) of Section 4. The actual states of equilibrium $\{p, S, P\}$ constitute cones, $\Phi(\rho, \beta, \cdot)$, $K(\rho, \beta)$, $E(\rho, \beta)$, respectively, which depend on the friction components ρ, β. From the correspondence rules (iii)–(v), Propositions 1 and 2, and Corollary 1 there follows, if $P \neq 0$:

Corollary 2. (i) Load P and displacement U are corresponding vectors if: $P \in E_n(\rho, \beta)$ and $U \in \Xi_b(\beta)$, or $P \in E_m(\rho, \beta)$ and $U = 0$, respectively. (ii) There is no correspondence between P and U if nonzero $P \in E_n(\rho, \beta)$ and $U \in \Xi_d(\beta)$, or $P \in E_m(\rho, \beta)$ and nonzero $U \in \Xi_b(\beta)$, or $P \in E_m(\rho_y, \beta)$ and $U \in \Xi_d(\rho, \beta)$, respectively.

The work equation for any AE $= \{P'', S'', p''\}^{\mathrm{T}}$ corresponding to friction $\tan \varphi$ and any admissible AK $= \{U', w', \gamma'\}^{\mathrm{T}}$ corresponding to inclination $\tan \beta$ is, according to eqns (16″) and (31),

$$\langle P'', U' \rangle = \langle |\sigma''|, |\gamma'_t| (\tan \varphi''_{\gamma'} - \tan \beta'_{\gamma'}) \rangle \leqslant \langle |\sigma''|, |\gamma'_t| (\tan \varphi_{\gamma'} - \tan \beta_\gamma) \rangle. \tag{37}$$

If $\{P, S, p\}^{\mathrm{T}}$ and $\{U, w, \gamma\}^{\mathrm{T}}$ are corresponding states, there holds

$$\langle P, U \rangle = \langle p, \gamma \rangle = \langle |\sigma|, |\gamma_t| (\tan \varphi_\gamma - \tan \beta_\gamma) \rangle \geqslant 0. \tag{37'}$$

The total dissipative work is for small γ

$$\Delta F(p, \gamma, \rho, \beta) = \int_{\cup \Gamma_v} \int_0^\gamma \mathbf{p} \cdot \mathrm{d}\boldsymbol{\gamma} \, \mathrm{d}\Gamma \cong \langle |\sigma|, |\gamma_t| (\tan \varphi_\gamma - \tan \beta_\gamma) \rangle. \tag{37''}$$

The equilibrium can be characterized by *energetic criteria*. If the structure is given at time t_0 an impulse, this induces a velocity field $\dot{u}$, a displacement field $\{\gamma, w, U\}^{\mathrm{T}}$ and an initial kinetic energy $T(t_0) = 1/2\Sigma\int \rho_d \dot{u}^2 \, \mathrm{d}\Omega > 0$. The energy balance after time interval Δt at constant loads P_v gives

$$\Delta E(\Delta t) = -\langle P, U \rangle + \Delta W + \Delta F + \Delta T = 0, \tag{38}$$

where ΔW, ΔF are the changes in the strain energy W and the dissipative potential F, respectively. Because of the rigidity of the bodies, $\Delta W = 0$. From eqn (36) we obtain kinematic criteria for equilibrium:

(a′) The equilibrium is stable if at given loads P for every nonzero $U' \in \Xi_a$, ΔT decreases. Hence

$$\langle P, U' \rangle < \Delta F(U'); \quad \forall \, U' \in \Xi_a(\beta). \tag{39}$$

(b′) The equilibrium is neutral if at a load P^c there is no field such that $\Delta T > 0$ but at least one admissible field u^c such that $\Delta T = 0$. Hence, in addition to eqn (39), there applies

$$\langle P^c, U^c \rangle = \Delta F(U^c). \tag{39'}$$

(c′) No equilibrium is possible (advancing collapse) if there is at least one field u_u for which ΔT increases

$$\langle P, U_u \rangle > \Delta F(U_u). \tag{39''}$$

The set of loads $\{P\} \subset Z$, which satisfy (a′) and (b′), forms the cone $E(\rho,\beta)$ in the sequel called the *cone of stability*. From (a′) and (b′) and Corollary 2 there follows:

Corollary 2′. The cone of neutral loads $E_c(\rho,\beta)$ equals the cone $E_n(\rho,\beta)$. Every admissible load P^c, to which correspond $U^c \in \Xi_b(\beta)$, is a neutral load.

The following general rules apply.

Proposition 3. In a non-monolithic assemblage without ties, no states of eigenstress p^o are possible. To every load $P = 0$, there corresponds the zero interface traction, $p^o \equiv 0$.

Proof. If $P = 0$, we suppose $p^o \neq 0$. For any U^a, γ^a and $p^o \in \Phi(\rho,\beta)$: $\langle p^o, \gamma^a \rangle_H = \langle 0, U^a \rangle_z = 0$. The negative polar cone of $\Phi(\rho,\beta)$ is $\Phi^*(\rho,\beta) = X'(\varphi) \subset X_a(\beta)$. The work equation for p^o and $\gamma' \in X'^o(\varphi)$, because of the convexity of the corresponding cone X'^o, gives: $\langle p^o, \gamma' \rangle < 0$ for $\forall\, \gamma' \in X'^o$, which leads to a contradiction if $p^o \neq 0$. □

Proposition 4. (i) On every interface $\Gamma_{\nu\mu}$, the normal resultant S_z is compressive. (ii) If the contour $C(s)$ of $\Gamma_{\mu\nu}$ is in the base plane $\pi^o_{\mu\nu}$ containing $\Gamma^o_{\mu\nu}$ and if $\tan\beta \tan\varphi < 1$, or if $\Gamma_{\nu\mu}$ is plane, the point $s_p \in \pi^o_{\nu\mu}$ of application of S_z, uniquely defined by the condition $\mathbf{M}(s_p) = M_z(s_p)\mathbf{k}$, is situated on the convex hull of $\Gamma^o_{\nu\mu}$, or $\Gamma_{\nu\mu}$, respectively.

Proof. (i) follows from the definition $S_z = \int \sigma \mathrm{d}\Gamma \leqslant 0$ since $\forall\, \sigma \leqslant 0$. (ii) The moment $\mathbf{M}(s)$ with respect to point $\mathbf{s} = x\mathbf{i} + y\mathbf{j} \in \Gamma^o_{\nu\mu}$ depends on the normal force $S_z\mathbf{k}$ and shear force $\mathbf{Q}$ on $\Gamma^o_{\nu\mu}$.

$$\mathbf{M}(s) = \mathbf{M}(0) - \mathbf{s} \times S_z\mathbf{k} - \mathbf{s} \times \mathrm{Q}; \quad (\mathbf{k} \perp \mathbf{s}).$$

If $\mathbf{s} = \mathbf{s}_p$, the component $\mathbf{M}_t(s_p)$ of $\mathbf{M}(s_p)$ in the plane $\pi^o_{\mu\nu}$ is zero

$$\mathbf{M}_t(s_p) = \mathbf{M}_t(0) - \mathbf{s}_p \times S_z\mathbf{k} = 0.$$

Hence $\mathbf{s}_p$ is uniquely determined by $\mathbf{M}(0)$ and S_z (Del Piero, 1989)

$$\mathbf{s}_p = \mathbf{M}(0) \times \mathbf{k} / |S_z|. \tag{40}$$

Choosing x in direction $\mathbf{M}_t(0)$, while retaining the origin 0 of $\mathbf{s}$ fixed within $\Gamma^o_\nu\mu$, we obtain

$$M_x(y_p) = \int_{\Gamma_{\mu\nu}} (y(s)\sigma - z(s)\tau_y)\,\mathrm{d}\Gamma - y_p S_z = 0$$

$$|y_p||S_z| = \left| \int_{\Gamma_{\mu\nu}} (y(s)\sigma - z(s)\tau_y)\,\mathrm{d}\Gamma \right| \leqslant \int_{\Gamma_{\mu\nu}} |\sigma|(|y(s)| + |z(s)|\tan\varphi)\,\mathrm{d}\Gamma$$

$$\leqslant |S_z| \sup_{s \subset \Gamma^o_{\mu\nu}} (|y(s)| + |z(s)|\tan\varphi)$$

because $|\tau_y| \leqslant |\sigma|\tan\varphi$. Since $|z_{,y}| \leqslant \tan\beta(s)$ on $\Gamma_{\nu\mu}$ and $z(c) = 0$ on the contour $C(s)$ there follows

$$|y_p| \leqslant \sup_{c \in C} |y(c) \tag{40}$$

if $|z(s)|\tan\varphi_y < |y(c)|_{\max} - |y(s)|$, which implies $\tan\beta \tan\varphi \leqslant 1$.

For any direction y, the inequality $y(s) \leqslant \sup y(c)$ represents a half-plane containing $\Gamma^o_{\nu\mu}$, bounded by a tangent to $\Gamma^o_{\mu\nu}$ at the outermost point of C in direction y. Since $0 \in \Gamma^o_{\nu\mu}$, the intersection of all these half-planes constitutes the convex hull of $\Gamma^o_{\nu\mu}$. □

7. NONDISSIPATIVE FRICTION

If the friction is nondissipative $\rho = 0$, $\varphi = \beta$, the cone $E_a(\varphi) = E_a(\beta)$, and the cones of stability in the spaces ∂H, ∂G and Z are $\Phi(0,\beta,\cdot)$, $K(0,\beta)$ and $E(0,\beta)$, respectively. The deformation cones are $X_a(\beta,\cdot)$, $\Lambda_a(\beta)$,$\Xi_a(\beta)$. In the case of nondissipative friction $\Delta F = 0$, the kinematic criteria (39) are:

(a″) stable equilibrium: $P'' \in E(0,\beta)$; $\langle P'', U' \rangle < 0$; $\forall U' \in \Xi_a(\beta)$; (41)

(b″) neutral equilibrium: $P^c \in E(0,\beta)$; $\langle P^c, U' \rangle \leqslant 0$; $U' \in \Xi_a(\beta)$;
with at least one pair corresponding P^c, U^c that satisfies $\langle P^c \cdot U^c \rangle = 0$; (41′)

(c″) no equilibrium: there is an excitation U_u such that $\langle P_u, U_u \rangle > 0$. (41″)

The work equations for admissible $\{P'', S'', p''\}^T$ and admissible $\{U', w', \gamma'\}^T$ are according to eqn (37)

$$\langle P'', U' \rangle = \langle p'', \gamma' \rangle_{\partial H} \leqslant 0; \quad \forall U' \subset \Xi_a(\beta). \tag{42}$$

For corresponding $\{P, S, p\}^T$ and $\{U, w, \gamma\}^T$ there applies, according to (37′),

$$\langle P, U \rangle_Z = \langle p, \gamma \rangle_{\partial H} = 0; \quad \langle S, w \rangle_{\partial G} = 0. \tag{42′}$$

The relations (42) and (42′) imply that any $P'' \in E_a(\beta)$ satisfies the stability conditions, hence $E(0,\beta) = E_a(\beta)$. They also express that $E_a(\beta)$ is contained in $\Xi^*(\beta)$, the polar cone of $\Xi_a(\beta)$.

Lemma 4. If the set of corresponding vectors $\{P^c, U^c\} = \{P = E_a(\beta), U \in \Xi_a(\beta) | \langle P, U \rangle = 0\}$ is non-empty: (i) the cones $E_a(\beta)$ and $\Xi_a(\beta)$ are polar cones

$$E_a(\beta) = \Xi_a^*(\beta); \quad \Xi_a(\beta) = E_a^*(\beta); \tag{42″}$$

(ii) P^c and U^c are corresponding generatrices of $E_a(\beta)$ and $\Xi_a(\beta)$, respectively.

Proof. (i) Let $\lambda > 0$ and $P_1 \in E_a^\circ(\beta), k \notin E_a(\beta)$. Since $E_a(\beta) \subset \Xi^*(\beta)$, suppose there is a $P(\lambda'') = P + \lambda'' k$ contained in $\Xi_a^*(\beta)$ but not in $E_a(\beta)$, and therefore satisfies $\langle P'', U' \rangle \leqslant 0$ for any $U' \in \Xi_a(\beta)$. Suppose $P(\lambda^c) = P_1 + \lambda^c k \in E_a(\beta)$ and $U^c \in \Xi_a(\beta)$ are corresponding vectors. Since $P'' \notin E_a(\beta)$, there holds $\lambda'' > \lambda^c$. Because $\langle P(\lambda^c), U^c \rangle = 0$, and choosing $\langle k, U^c \rangle > 0$, we obtain $\langle P(\lambda''), U^c \rangle = (\lambda'' - \lambda^c)\langle k, U^c \rangle > 0$. This contradicts the assumption that $P(\lambda'') \in \Xi_a^*(\beta)$. Therefore $\Xi_a^*(\beta)$ contains only loads $P^a \in E_a(\beta)$.

(ii) Let $U_1 \in \Xi_a^\circ(\beta)$, $q \notin \Xi_a(\beta)$ and $U(\mu) = U_1 + \mu q$, and define $P(\lambda) = P_1 + \lambda k$ according to (i). If $\langle P(\lambda^c), U(\mu^c) \rangle = 0$ we can choose k and q so that $\langle P(\lambda^c), q \rangle > 0$, $\langle k, U(\mu^c) \rangle > 0$ with $\lambda, \mu > 0$. The associated generatrices of $E_a(\beta)$ and $\Xi_a(\beta)$ are $P(\lambda^g)$ and $U(\mu^g)$ where $\lambda^g \geqslant \lambda^c$, $\mu^g \geqslant \mu^c$, respectively. If we choose λ and μ within the admissible intervals $[\lambda^c, \lambda^g]$ and $[\mu^c, \mu^g]$, respectively, there holds

$$\langle P(\lambda)\, U(\mu^c) \rangle = (\lambda - \lambda^c)\langle k, U(\mu^c) \rangle \geqslant 0$$

$$\langle P(\lambda^c), U(\mu) = (\mu - \mu^c)\langle \mathrm{P}(\lambda^c), q \rangle \geqslant 0.$$

Hence vectors $P(\lambda) \notin E_a(\beta)$; $U(\mu) \notin \Xi_a(\beta)$ if $\lambda^c \neq \lambda^g$, $\mu^c \neq \mu^g$. Correspondence is possible only if $\lambda^c = \lambda^g$, $\mu^c = \mu^g$ and $P(\lambda^c)$ and $U(\lambda^c)$ are corresponding generatrices of the lateral surfaces $\partial E_a(\beta)$ and $\partial \Xi_a(\beta)$, respectively. □

From this there follows:

Proposition 5. If the friction is nondissipative, $\varphi = \beta$, the cone of stability $E(0, \beta)$: (i) equals the cone of admissible loads $E(0, \beta) = E_a(\beta)$; (ii) is the negative polar cone of the cone of admissible displacements; $E(0, \beta) = \Xi_a^*(\beta)$; (iii) the set of neutral loads $E_n(0, \beta) = \partial E_a(\beta)$ is closed with empty interior, $E_n^0(0, \beta) = 0$; any neutral load $P^c \in E(0, \beta)$ is a generatrix P^g of $\partial E_a(\beta)$, to which corresponds a generatrix $U^c \in \partial \Xi_a$ orthogonal to the supporting hyperplane of $E(0, \beta)$, containing P^c; (iv) the cone of mechanisms $\Xi_b(\beta)$, equals $\partial \Xi_a(\beta)$; (v) the set of stable loads is the interior $E^0(0, \beta)$ of $E_a(\beta)$.

Corollary 5. If the friction is nondissipative the relation

$$\langle P, U' \rangle \leqslant 0; \quad \forall U' \in \Xi_a(\beta) \tag{43}$$

is a sufficient and necessary condition for equilibrium at load P.

Proof. If the assemblage is in equilibrium for any excitation U', there holds, according to eqn (42), $\langle P, U' \rangle \leqslant 0$. Hence, eqn (43) is a necessary condition for equilibrium. If in the initial state the assemblage is at rest and there is *no* equilibrium, the kinetic energy increases from 0 to $\Delta T > 0$ because of the occurring acceleration. Therefore $\langle P, U' \rangle > 0$, according to eqn (41″). Hence eqn (43) is a sufficient condition for equilibrium. □

An arbitrary load $P \in E(0, \beta)$ may be expressed by the vector $P = P^1 + \lambda k$ where $P^1 \in E^0(0, \beta)$ is a fixed stable load, $k \notin E_a(\beta)$ is a direction vector, and λ is the load intensity of k. P^1 and k determine a plane $R^2 \subset Z$ because λk intersects $\partial E(0, \beta)$ in two points $P_+ = P^1 + \lambda_+ k$ and $P_- = P^1 + \lambda_- k$ where $\lambda_+ > 0$, $\lambda_- < 0$ (Luenberger, 1968). The stable range corresponding to P^1 and k is then determined by the limit loads $P_{c+} = P^1 + \lambda_+{}^c k$; and $P_{c-} = P_1 + \lambda_-{}^c k$, and the corresponding vertex angle of $E(0, \beta)$ or *angle of stability* θ in direction k

$$\theta(k) = \widehat{P_c, P_{c+}}. \tag{44}$$

Because $0 \leqslant \theta(\mathrm{k}) \leqslant \pi$, θ increases monotonously with $(\lambda_+^c - \lambda_-^c)$. The range $(\lambda_-^c, \lambda_+^c)$ of stable loads may be defined, either *kinematically* by a state AK with $\gamma'(\cdot) \in X_a(\beta, \cdot)$, $U' \in \Xi_a(\beta)$ and P' U', hence

$$\langle P' \, U' \rangle = \langle P^1, U' \rangle + \lambda_\varepsilon' \langle k, U' \rangle = 0 \tag{45}$$

which provides the intensity observing $\langle P^1, U' \rangle < 0$

$$\lambda_\varepsilon' = |\langle P^1, U' \rangle| / \langle k, U' \rangle, \tag{45'}$$

or *statistically* by an admissible state of equilibrium $\{p'', S'', P''\}^{\mathrm{T}}$ where

$$P'' = P^1 + \lambda_\sigma'' k \in E_a(\beta); \quad p''(\cdot) \in \Phi_a(\beta, \cdot). \tag{46}$$

θ can then be determined by the intervals $\lambda_{\varepsilon+}' - \lambda_{\varepsilon-}'$ and $\lambda_{\sigma+}'' - \lambda_{\sigma-}''$.

Proposition 6. If the friction is nondissipative $\varphi = \beta$, the vertex angle θ of the stability cone $E(0, \beta)$ for load P^1 and direction k is:

(a) defined for all admissible excitations $\{\gamma', w', U'\}^{\mathrm{T}}$, by $\theta_\varepsilon' = \theta(\lambda_{\varepsilon+}', \lambda_{\varepsilon-}')$ attains its minimum in the actual neutral state $\{P^1 + \lambda^c k, U^c\}^{\mathrm{T}}$, which corresponds to an AE where $p^c(\cdot) \in \partial \Phi_a(\beta, \cdot)$, (principle of minimum angle of stability);

(b) defined for all admissible states $\{p'', S'', P''\}^{\mathrm{T}}$ by $\theta_\sigma'' = \theta(\lambda_{\sigma+}'', \lambda_{\sigma-}'')$ attains its maximum in the actual neutral state $\{P^1 + \lambda^c k, U^c\}^{\mathrm{T}}$, which corresponds to an AK where $\gamma^c(\cdot) \in X_a(\beta, \cdot)$, (principle of maximum angle of stability).

Proof. (a) is proved by subtracting from eqn (45) the work expression for $\{U', \gamma'\}^T$ and the actual $\{P^1+\lambda^c k, p^c\}^T$: $\langle P^1 \cdot U'\rangle + \lambda^c\langle k, U'\rangle = \langle p^c, \gamma'\rangle \leqslant 0$. Therefore $(\lambda' - \lambda^c)\langle k, U'\rangle = -\langle p^c, \gamma'\rangle \geqslant 0$; hence $\lambda'_+ - \lambda^c_+ \geqslant 0$ $\lambda'_- - \lambda^c_- \leqslant 0$ and $\lambda'_+ - \lambda'_- \geqslant \lambda^c_+ - \lambda^c_-$. (b) is proved applying the work expression of the actual state $\{U^c, \gamma^c\}^T$ to the AE states $\{P^1+\lambda^c k, p^c\}^T$ and $\{P^1+\lambda'' k, p''\}^T$, respectively:

$$\langle P^1, U^c\rangle + \lambda^c\langle k, U^c\rangle = \langle p^c, \gamma^c\rangle = 0$$

$$\langle P^1, U^c\rangle + \lambda''\langle k, U^c\rangle = \langle p'', \gamma^c\rangle \leqslant 0.$$

Subtraction gives $(\lambda^c - \lambda'')\langle k, U^c\rangle \geqslant 0$.

Geometrically, they follow immediately from the etremum characteristics of the cones $E_a(\beta)$ and $\Xi_a(\beta)$.

(a) $E(0, \beta)$ is the smallest polar cone Ξ'^* of any Ξ' (β) defined by intrapyramidal vectors $U' \in \Xi_a(\beta)$;

(b) $E(0, \beta)$ is the largest cone contained in $E_a(\beta)$. Hence $E''(0, \beta) \subset E_a(\beta)$. □

Although generators P^c of the boundaries $\partial E(0, \beta)$ thus are uniquely determined, the interface stress function $p(\cdot)$ in the neutral state remains undetermined because the mapping $BT: \partial H \to Z$ is not bijective.

Corollary 6. If in an assemblage of rigid bodies the friction is nondissipative and at a given load P a state of maximum dynamic contact is possible, the load P is stable.

Proof. Since in this case the interface stress function $p'''(\cdot) \in \Phi_m(\beta, \cdot) \subset \Phi^o_a(\beta, \cdot)$, there applies for every admissible excitation $\{U', \gamma'\}^T$: $\langle P, U'\rangle = \langle p''', \gamma'\rangle < 0$. Hence for any p''' and any U', P is stable. Supposing there is a corresponding admissible function $p'' \notin \Phi'''(\beta)$ and an excitation $\{U'', \gamma''\}$ where $\langle P, U''\rangle = \langle p'', \gamma''\rangle = 0$, this would imply that P is not stable. But this contradicts the condition that $\langle P, U''\rangle = \langle p''', \gamma''\rangle < 0$. □

8. NONDISSIPATIVE FRICTION WITH DISSIPATIVE FRICTION

In this case ρ and $\beta > 0$. The friction cone with $\varphi = (\rho+\beta)_\rho$ is $\Phi(\rho, \beta, \cdot)$ and the deformation cone is $X(\beta, \cdot) = X_a(\beta, \cdot)$. From Corollary 1 and Proposition 5 (iv) there follows

Lemma 5. The set of mechanisms $\Xi_b(\beta)$ corresponding to the set of neutral loads $E_n(\rho, \beta)$ is independent of ρ. Hence $\Xi(\rho, \beta) = \Xi_a(\beta)$.

From eqns (37) and (37″) and the correspondence rules (iii) and (v) of Section 4 we obtain:

Lemma 6. A necessary condition for sliding contact and hence for the occurrence of dissipative work ΔF is: nonzero $\gamma(\cdot) \in \partial X(\beta, \cdot)$; nonzero $p(\cdot) = \partial\Phi(\rho, \beta, \cdot)$. If either $\gamma(\cdot) \in X^o(\beta, \cdot)$ or $p(\cdot) \in \Phi^o(\rho, \beta, \cdot)$, no sliding with contact is possible and $\Delta F = 0$.

Propositions concerning the uniqueness of the neutral load P^c and the cone of stability $E(\rho, \beta)$ can be derived analogously to those concerning nondissipative friction. Thus if $P^1 \in E^o(\rho, \beta)$ and $k \notin E(\rho, \beta)$ with $P = P^1 + \lambda k \in \partial E(\rho, \beta)$, we define λ either kinematically by an AK state $\{U', w', \gamma'\}^T$

$$\lambda'_c = (\langle |\sigma'|, |\gamma'|(\mathrm{tg}\varphi_{\gamma'} - \mathrm{tg}\beta_{\gamma'})\rangle_{\partial H} + |\langle P^1, U'\rangle_z|)/\langle k, U'\rangle \tag{45″}$$

or statically by an AE state $\{P^1 + \lambda''_\sigma k, S'', p''\}^T$. By using the convexity and sectional normality rules, we obtain by proceeding as in Section 7:

Proposition 7. If the normal stress σ in every joint $\Gamma_{\nu\mu}$ is determined only by P^1, then the vertex angle $\theta(\rho,\beta)$ of $E(\rho,\beta)$ corresponding to a load $P+\lambda k$ is:

(a) defined for all permissible excitations $\{\gamma', w', U'\}^T$ by $\theta'_\varepsilon = \theta(\lambda'_+, \lambda'_-)$, attains its minimum in the actual limit state (P^c, U^c);
(b) defined for all permissible states $\{p'', S'', P''\}^T$ by $\theta''_\sigma = \theta(\lambda''_+, \lambda''_-)$, attains its maximum in the actual limit state (P^c, U^c).

Because the restrictions imposed on σ only in special cases (see Section 9) are satisfied, there are generally no unique bounds of the neutral state. On the other hand, Propositions 5 and 6 for nondissipative friction provide well-defined bounds for the cone $E(\rho,\beta)$.

Proposition 8. The vertex angle $\theta(\rho,\beta)$ for load $P^1 \in E^\circ(\rho,\beta)$ and any direction $k \in E(\rho,\beta)$ is enclosed in the interval $[\theta(0,\beta), \theta(0,\rho+\beta)]$ with $\theta(0,\beta) < \theta(\rho,\beta) < \theta(0,\rho+\beta)$. Hence

$$E(0,\beta) \subset E(\rho,\beta) \subset E(0,\rho+\beta); \quad \varphi = \rho+\beta. \tag{47}$$

Proof. The last inclusion $E(\rho,\beta) \subset E(0,\rho+\beta)$ follows from $E(0,\rho+\beta) = E_a(\rho+\beta)$. Let $P^c(\varphi) = P^1 + \lambda^c(\varphi)k \in \partial E(\rho,\beta)$ with the corresponding $U^c(\varphi) \in E^*(\rho,\beta) \subset \Xi(\beta)$, and $P^c(\beta) = P^1 + \lambda^c(\beta)k \in \partial E(0,\beta)$ with the corresponding $U^c(\beta) \in \Xi(\beta)$. Writing the work equations

$$\langle P^1 + \lambda^c(\varphi)k, U^c(\varphi)\rangle = \langle |\sigma^c|, \ |\gamma^c_t|(\tan\varphi - \tan\beta)\rangle \geqslant 0, \quad \langle P^1 + \lambda^c(\beta)k, U^c(\varphi)\rangle \leqslant 0.$$

Substraction gives $(\lambda^c_i(\varphi) - \lambda^c_i(\beta))\langle k, U^c(\varphi)\rangle \geqslant 0$. Hence $E(0,\beta) \subset E(\rho,\beta)$ □

In the sequence (47) only $E(0,\beta)$ and $E(0,\rho+\beta)$ are well defined, whereas for $E(\rho,\beta)$ the inclusions give rather poor estimates because of the indefiniteness of the neutral state. If contact sliding is excluded, $\gamma_t = 0$, the remaining possible mechanism $\{U^h, w^h, \gamma^h\}^T$ is a pure hinge mechanism. The corresponding Ξ^h is independent of ρ and β, being a proper subset $\Xi_a(\pi/2)$ of $\Xi_a(\beta)$. Since, according to Proposition 5, $E_a(\pi/2) = \Xi^*_a(\pi/2)$, and $\Xi_a(\pi/2) \subset \Xi(\beta)$, we get the inclusion

$$E(0,\rho+\beta) \subset E_a(\pi/2). \tag{48}$$

Although the hinge mechanism is independent of the friction φ, the transition from hinging to sliding depends on a critical angle φ_h which depends on the direction of the load P and the interface forces S.

According to eqns (25) and (48), the condition that no contact sliding is possible at P, with given φ, β is

$$p''_s \in (BT)^{-1}P \cap \Phi(\varphi,\cdot); \quad \langle P, U'\rangle < \langle |\sigma''_s|, |\gamma'_t|(\tan\varphi_{\gamma'} - \tan\beta_{\gamma'})\rangle; \quad \forall \gamma' \in X_a(\beta),\, U' \in \Xi_a(\beta).$$

As an indicator of contact sliding, we define at given P and φ, β

$$\psi(P, U', \varphi, \beta) = \frac{\langle |\sigma''|, |\gamma'_t|(\tan\varphi_{\gamma'} - \tan\beta_{\gamma'})\rangle}{\langle P, U'\rangle}, \tag{49}$$

where $p''(\cdot) = (TB)^{-1}P \cap \Phi(\varphi,\cdot) \subset \Phi_a(\rho,\beta,\cdot)$ and $\gamma'(\cdot) = T'B'U' \cap X(\beta,\cdot) \in X_a(\beta,\cdot)$, with $U' = \{U' \in \Xi_a(\beta) \,|\, \langle P, U'\rangle > 0\}$.

In order to classify sliding at given φ, we define

$$\psi_i(P,\varphi,\beta) = \inf_{p'',U'} \frac{\langle |\sigma''|, |\gamma'_t|(\tan\varphi_{\gamma'} - \tan\beta_{\gamma'})\rangle}{\langle P, U'\rangle} = \frac{\langle |\sigma|, |\gamma_t|(\tan\varphi_\gamma - \tan\beta_\gamma)\rangle}{\langle P, U\rangle} \geqslant 0 \tag{49'}$$

as an indicator of safety, where $\psi_i = 1$ stands for contact sliding. If $\psi_i < 1$, no equilibrium is possible; if $\psi_i > 1$, no contact sliding is possible. If $\psi_i(P, \varphi, \beta) = 1$ and suppose $\psi(P, U', \varphi', \beta) = 1$ where $\varphi' > \varphi$, then, observing eqn (37), we obtain

$$1 = \frac{\langle |\sigma''|, |\gamma_t'|(\tan \varphi_{\gamma'}' - \tan \beta_{\gamma'})\rangle}{\langle P, U' \rangle} > \frac{\langle |\sigma''|, |\gamma_t'|(\tan \varphi_{\gamma'} - \tan \beta_{\gamma'})\rangle}{\langle P, U' \rangle} \geqslant \psi_i(P, \varphi, \beta)$$

which contradicts $\psi_i(P, \varphi, \beta) = 1$. Hence φ is the upper bound $\varphi_{\max}$ at which contact sliding can occur. Defining in the same way

$$\psi_{SU}(P, \varphi, \beta) = \sup_{p'', U'} \frac{\langle |\sigma''|, |\gamma_t'|(\tan \varphi_{\gamma}' - \tan \beta_{\gamma}')\rangle}{\langle P, U' \rangle} = \frac{\langle |\sigma|, |\gamma_t|(\tan \varphi_{\gamma} - \tan \beta_{\gamma})\rangle}{\langle P, U \rangle}, \qquad (49'')$$

we can show that for a given P there is a $\varphi_{\min}$ such that no equilibrium with contact sliding is possible at any value φ' lower than $\varphi_{\min}$. From eqn (37) there follows $\psi_{SU}(P^a, \varphi, \beta) \geqslant 1$.

At a given load P neutral states of sliding may thus materialize at different friction angles φ within an interval $[\varphi_{\min}(P), \varphi_{\max}(P)]$. If $\varphi > \varphi_{\max}(P)$, then $\psi_i(P, \varphi, \beta) > 1$ and P is certainly outside the range of neutral loads E_n and hence stable. The set of neutral loads $E_n(\rho, \beta)$ is thus not confined to a boundary $\partial E_a(0, \beta)$ with an empty interior in Z, as in the case of nondissipative friction, but constitutes a non-convex closed cone with a non-empty interior in the load space. The load P is *certainly safe* at the friction angles φ, β if $\varphi = \rho + \beta > \varphi_{\max}(P)$, which is equivalent to the condition $\psi_i(P, \varphi, \beta) > 1$. The set of certainly safe loads $E_s^0(\rho, \beta)$, thus defined, is a subset of the set of equilibrium loads $E(\rho, \beta)$.

Proposition 9. The set of certainly safe loads constitutes the interior $E_s^0(\rho, \beta)$ of the closed convex cone $E_s(\rho, \beta)$ defined by the safety indicator $\psi_i(P, \varphi, \beta) = 1$. The genetratrices of $E_s(\rho, \beta)$ constitute the closure of $E_s^0(\varepsilon, \beta)$ and represent the neutral loads $P^c \in E_n(\rho, \beta)$ corresponding to $U^c \in \partial \Xi_a(\beta)$. The set $\{P\}$ defined by $\psi_{SU}(P, \varphi, \beta) \geqslant 1$ equals $E_a(\varphi)$.

Proof. The conical shape and the closeness follows from the definition of ψ_i. If P_1 and P_2 are certainly safe loads with $\psi_i(P_1, \varphi, \beta) > 1$, $\psi_i(P_2, \varphi, \beta) > 1$, the convexity depends on the safety of $P_1 + P_2$. If (p_1'', P_1) and (p_2'', P_2) are AE and $(\gamma', ')$ is AK we obtain, taking into account eqns (49) and (49') and $\psi(P, U', \varphi, \beta) \geqslant \psi_i(P, \varphi, \beta)$,

$$\langle |\sigma_1''|, |\gamma'|(\tan \varphi_{\gamma'} - \tan \beta_{\gamma'})\rangle > \langle P_1, U' \rangle; \quad \forall p_1'', \forall U'$$

$$\langle |\sigma_2''|, |\gamma'|(\tan \gamma_{\gamma'} - \tan \beta_{\gamma'})\rangle > \langle P_2, U' \rangle; \quad \forall p_2'', \forall U'.$$

$$\langle |\sigma_1'' + \sigma_2''|, |\gamma_t'|(\tan \varphi_{\gamma'} - \tan \beta_{\gamma'})\rangle > \langle P_1 + P_2, U' \rangle; \quad \forall p'' = p_1'' + p_2'', \forall U'. \qquad (50)$$

If $\psi_i(P_1 + P_2, \varphi, \beta)$ corresponds to admissible γ, U and admissible p, then

$$\langle |\sigma|, |\gamma|(\tan \varphi_{\gamma} - \tan \beta)\rangle = \psi_i(P_1 + P_2, \varphi, \beta) \langle P + P_2, U \rangle. \qquad (50')$$

The transition $|\sigma_1'' + \sigma_2''| \to |\sigma|$ is admissible because of the great indefiniteness of the certainly stable equilibrium of rigid body assemblages.

According to inequality (50) the left side must exceed $\langle P_1 + P_2, U \rangle$. Hence $\psi_i(P_1 + P_2, \varphi, \beta) > 1$, and $P_1 + P_2$ is a certainly safe load. This implies that the set of certainly safe loads $E_s^0(\rho, \beta)$, is an open convex cone. Repeating now the same procedure for loads P_1 and P_2 where $\psi_i(P, \varphi, \beta) = 1$ and $\psi_i(P, \varphi, \beta) = 1$ we can show that the cone $E_s(\rho, \beta)$ is convex. Since $E_s^0(\rho, \beta)$ does not contain loads for which $\psi_i(P, \varphi, \beta) = 1$ the loads P^c to which correspond $U^c \subset \Xi_a(\beta)$ are contained in the cone $E_n(\rho, \beta)$ outside $E_s^0(\rho, \beta)$. Proceeding as in Section 7 (Lemma 3), we can for every P^1 and direction k determine a smallest $|\lambda^c|$ which define a neutral load $P^c \in \partial E_s(\rho, \beta)$ corresponding to $U^c \in \partial \Xi_a(\beta)$ according to Lemma 4. □

The indicator of safety ψ_i makes it possible to estimate the degree of stability of the cones $E_s(\rho+\beta, 0)$, $E_s(\rho, \beta)$, $E(0, \rho+\beta)$, which all have the same resulting friction angle $\varphi = \rho+\beta$. Let sliding γ^c occur at P if $\beta = 0$ with $\psi_i(P, \varphi, 0) = 1$, where $\gamma^c(0, \cdot) \in \partial X(0, \cdot)$ and $P \in \partial E(\varphi, 0)$. Since $\gamma(\beta, \cdot) \in X_a(\beta, \cdot) \subset X_a^0(0, \cdot)$, no contact sliding, according to Lemma 5, is possible for $\gamma(\beta, \cdot) \in X_a(\beta, \cdot)$. Hence $\psi_i(P, \varphi, \beta) > 1$, which means that the cone $E_s(\rho, \beta)$ is safer than $E_s(\rho+\beta, 0)$ or

$$E_s(\rho+\beta, 0) \subset E_s(\rho, \beta). \tag{51}$$

From inclusions (47), (48) and (51), by varying ρ and β, we arrive at the following rule:

Proposition 10. The stability of a non-monolithic assemblage without ties increases monotonously and independently with each component ρ and β of friction, according to the sequence of inclusions

$$E(0,0) \subset E_s(\beta, 0) \subset E_s(\rho+\beta, 0) \subset E(\rho, \beta) \subset E(0, \rho+\beta) \subset E(\pi/2); \quad \tan\rho\tan\beta < 1. \tag{52}$$

On the contrary, the set of neutral states $E_n(\rho, \beta) = E_a(\rho+\beta) - E_s^0(\rho, \beta)$, increases with the dissipativety. Since the interiors of $E_n(0,0)$ and $E_n(0, \rho+\beta)$ are empty too, eqn (52) corresponds to

$$\varnothing = E_n^\circ(0,0) = E_n^\circ(0, \rho+\beta) \subset E_n^\circ(\rho, \beta) \subset E_n^\circ(\rho+\beta, 0). \tag{52'}$$

The determination of the safe cones E_s is dependent on the $6n$ degrees of freedom of the structure. A low number of possible mechanisms greatly facilitates the solution of the

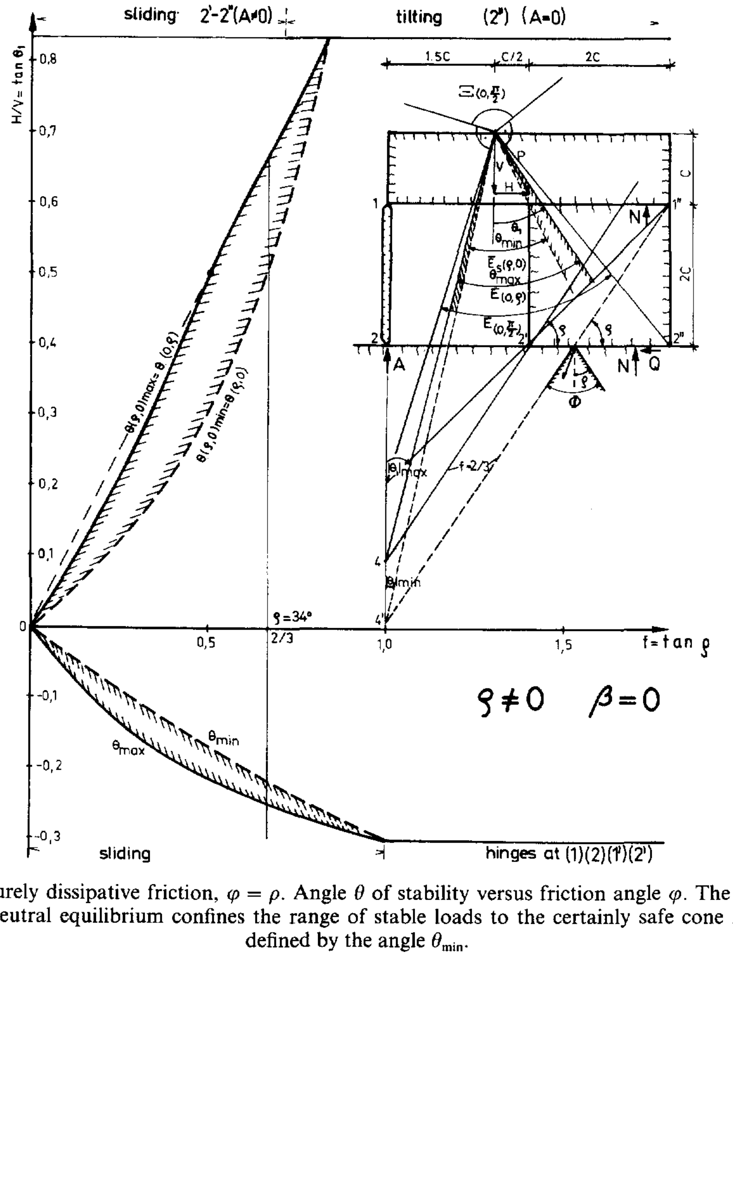

Fig. 4. Purely dissipative friction, $\varphi = \rho$. Angle θ of stability versus friction angle φ. The shaded area of neutral equilibrium confines the range of stable loads to the certainly safe cone $E_s(\rho, 0)$ defined by the angle $\theta_{\min}$.

problem. Especially in plane problems, the cone E_s can be determined simply by graphical means (Fig. 4). This method is unfortunately not applicable to three-dimensional structures. The effect of contact sliding is, because of twisting in the interfaces, much more pronounced in these structures.

9. THREE-DIMENSIONAL STRUCTURES WITH PLANE INTERFACES

We assume that the friction φ, β is isotropic and the interface $\Gamma_{0\mu}$ is in a plane π with $\mathbf{s} = x\mathbf{i}+y\mathbf{j}$ in a local system x, y. The impenetrability condition (12′) : $\gamma_n \geqslant |\gamma_t| \tan\beta$, provides the means to determine the range of stable loads on $\Gamma_{0\mu}$. A deformation field $\boldsymbol{\gamma}_{\mu 0} = \mathbf{v}_\mu + \boldsymbol{\omega}_\mu \times \mathbf{s}$, where $\mathbf{v}_\mu = \mathbf{v}_t + \mathbf{k}v_z$; $\boldsymbol{\omega}_\mu = \boldsymbol{\omega}_t + \mathbf{k}\boldsymbol{\omega}_z$, has components

$$\gamma_t = \boldsymbol{\omega}_z \mathbf{k} \times (\mathbf{s}-\mathbf{s}_F)\,; \quad \gamma_n \mathbf{k} = v_z\mathbf{k} + \boldsymbol{\omega}_t \times (\mathbf{s}-\mathbf{s}_F), \tag{53}$$

where $\mathbf{v}_t = \omega_z\mathbf{k} \times (\mathbf{s}-\mathbf{s}_F)$ and $\mathbf{s}_F$ is the centre of twist F. Placing $\mathbf{s}_F$ at the origin and the x axis in direction $\boldsymbol{\omega}_t$, with $\boldsymbol{\gamma}_t = \omega_z\mathbf{k} \times \mathbf{s}$; $\gamma_n = v_z + \omega_x y$, and $r = |\mathbf{s}-\mathbf{s}_F| = (x^2+y^2)^{1/2}$, the impenetrability condition (12′) can be written as

$$v_z + \omega_x y \geqslant |\omega_z| r \tan\beta. \tag{54}$$

The gap

$$g(x, y) = v_z + \omega_x y - |\omega_z| r \tan\beta \tag{55}$$

partitions the plane into the permissible domain Γ_+ where $g \geqslant 0$ and the forbidden domain with interpenetration Γ_-, where $g < 0$. Geometric, and hence dynamic, contact can occur only on the curve $g(x, y) = 0$ which can be written as

$$v_z/|\omega_z| + (\omega_x/|\omega_z|)y = r \tan\beta. \tag{56}$$

The left side represents an elevated plane $z = z_o + \alpha y$, whereas the right side represents a circular cone. The locus of contact $g(x, y) = 0$ is the projection of the intersection curve of the cone and the plane z on the base plane. The curve g depends on the inclination, $\alpha = \omega_x/|\omega_z|$ and the translation $z_o = v_z/|\omega_z|$. Therefore (Figs 5 and 6):

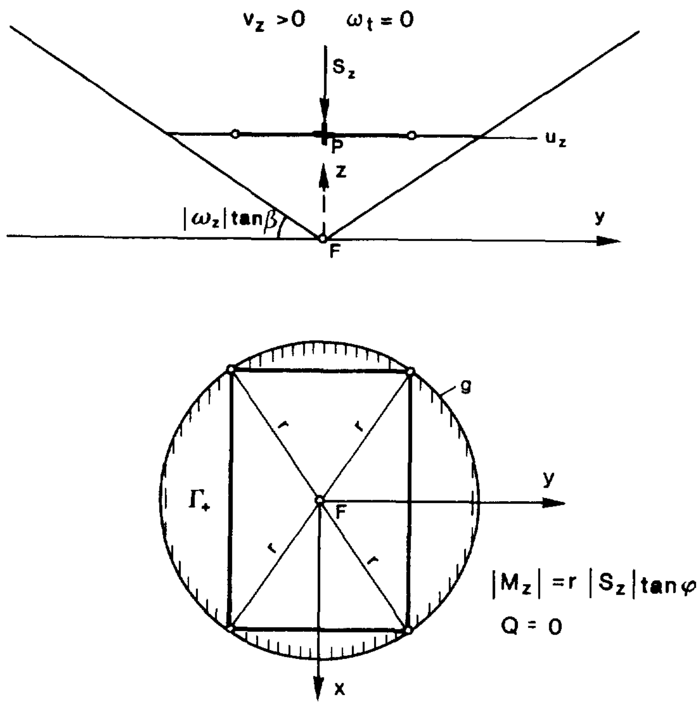

Fig. 5. Loci of contact $g(x,y)$: (a) four-point circular contact; (b) four-point elliptic contact; — contour of interface; ○ point of contact; shaded area Γ_+

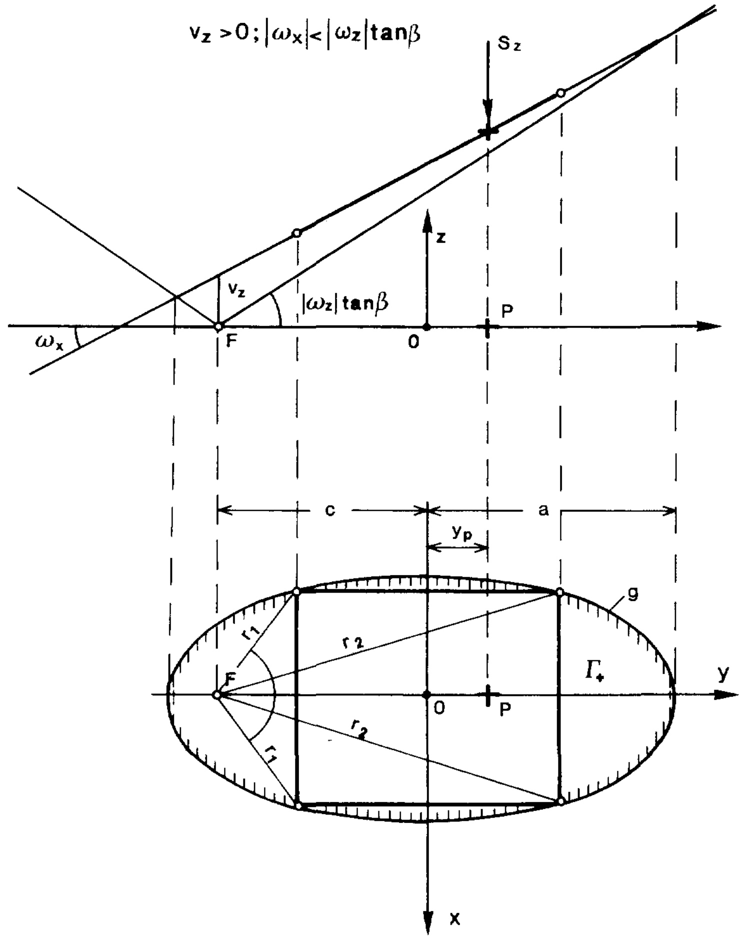

Fig. 5. (b).

(i) if $z_o = v_z/|\omega_z| \neq 0$, the curves are either circles ($\alpha = 0$), ellipses ($\alpha < \mathrm{tg}\beta$), a parabola ($\alpha = \mathrm{tg}\beta$), or hyperbolas ($\alpha > \mathrm{tg}\beta$) with the focus $\mathbf{s}_F \in \Gamma_+$ if $z_o > 0$, and $\mathbf{s}_F \in \Gamma$ if $z_o < 0$, respectively;
(ii) if $z_o = 0$ and $\alpha < \tan\beta$, the only point of possible contact is the focus; if $|\alpha| = \tan\beta$, the parabola degenerates into an arrow in direction y, finally if $|\alpha| > \tan\beta$ the hyperbola degenerates into two asymptote arrows from the origin;
(iii) if $\omega_z = 0$; $\omega_x \neq 0$ the curve degenerates into a straight line and Γ_+ is a half-plane;
(iv) the position $\mathbf{s}_F$ of the focal point F_i determines the semiaxes a', b' and principal axes x', y' of the admissible region Γ'_{+i}. The contour of $\Gamma_{o\mu}$ remains always within any Γ'_{+i}:

$$\Gamma_{o\mu} \in \cap\, \Gamma'_{+i} = \mathrm{conv}\, \Gamma_{o\mu} \text{ because of (iii).} \tag{57}$$

Proposition 11. If the friction is isotropic with component $\beta > 0$, and on interface $\Gamma_{0\mu}$ is imposed an excitation $\{v,\omega\}^T \in \partial\Lambda_{o\mu}$, it induces a deformation field γ where the loci of possible contact $g(x,y) = 0$ are conical sections or their asymptotes. The actual contact is confined to extreme points or such parts of the contour $\partial\Gamma_{o\mu}$ that coincide with the curve $g = 0$. The pole $\mathbf{s}_F$ of twisting ω_z constitutes the focus of the conical section. If $\mathbf{s}_F \to \infty$, then $\omega_t, \omega_z - > 0$ and the deformation is reduced to constant tangential and normal translation ($\gamma_t = v_t$; $\gamma_n = |v_t| \tan\beta$) with contact over the whole area of $\Gamma_{0\mu}$. The region of contact is dependent only on β and not on ρ.

Choosing $\boldsymbol{\omega}_t = \omega_x \mathbf{i}$ in the direction of a principal axis the conic $g(x,y) = 0$ becomes

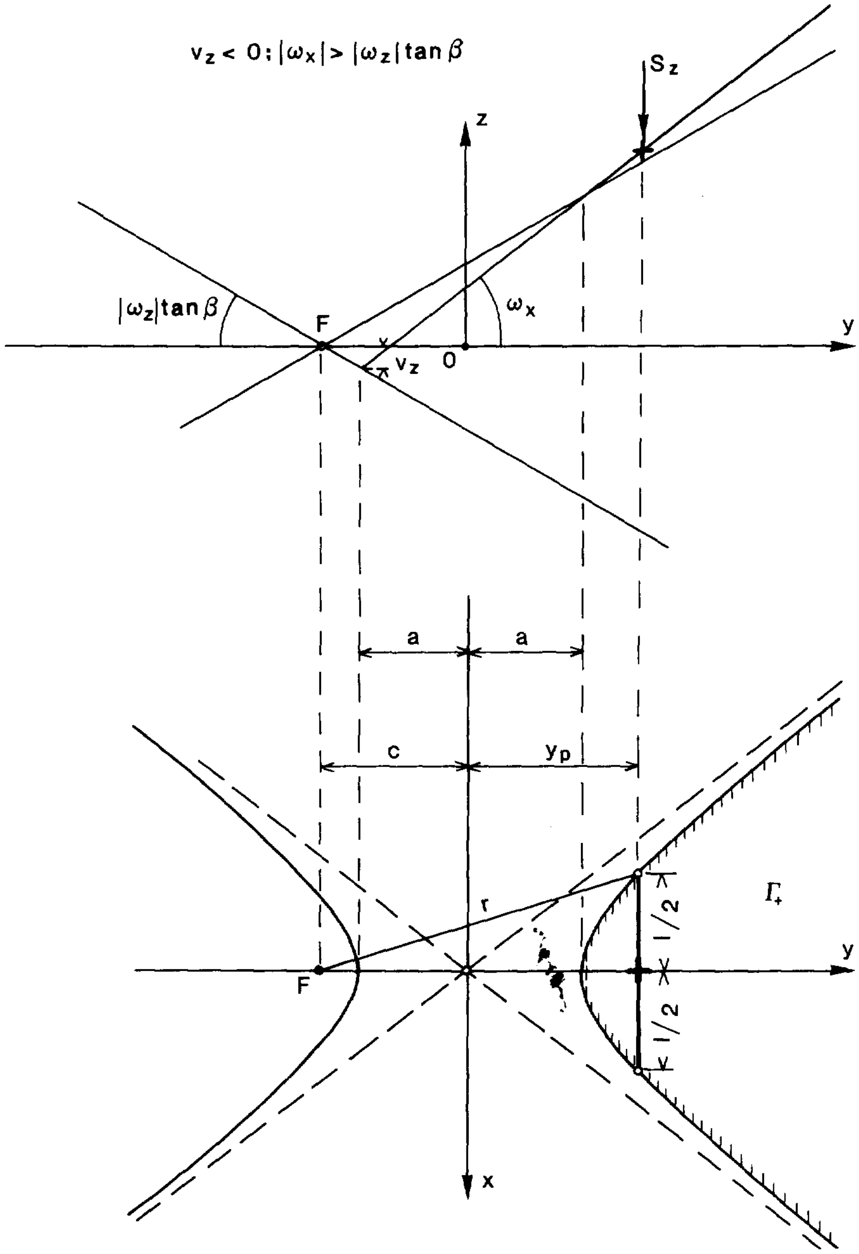

Fig. 6. Loci of contact: (a) Two-point hyperbolic contact; (b) Asymptotic contact with section and point; = line segment of contact; ○ point of contact.

$$(y^2/a^2) \pm (x^2/b^2) = 1. \tag{58}$$

$c^2 = a^2 \mp b^2$ defines the focal distance and $r = a + \varepsilon y$ the focal radius with $\varepsilon = c/a$, where + stands for ellipse and − stands for hyperbola with the asymptotes $y = \pm(a/b)x$ (Figs 5 and 6). The parameters a, b, c, ε are determined by the mechanism $\{\mathrm{v},\omega\}^{\mathrm{T}} \in \Lambda(\beta)$.

$$a = \frac{|v_z(0)|}{|\omega_z|\tan\beta\,(1-(\omega_x/\omega_z\tan\beta)^2)}; \quad b = \frac{|v_z(0)|}{|\omega_z|\tan\beta\,(1-(\omega_x/\omega_z\tan\beta)^2)^{1/2}};$$

$$c = \frac{|\omega_x|}{|\omega_z|\tan\beta}a; \quad \varepsilon = \frac{|\omega_x|}{|\omega_z|\tan\beta} \tag{59}$$

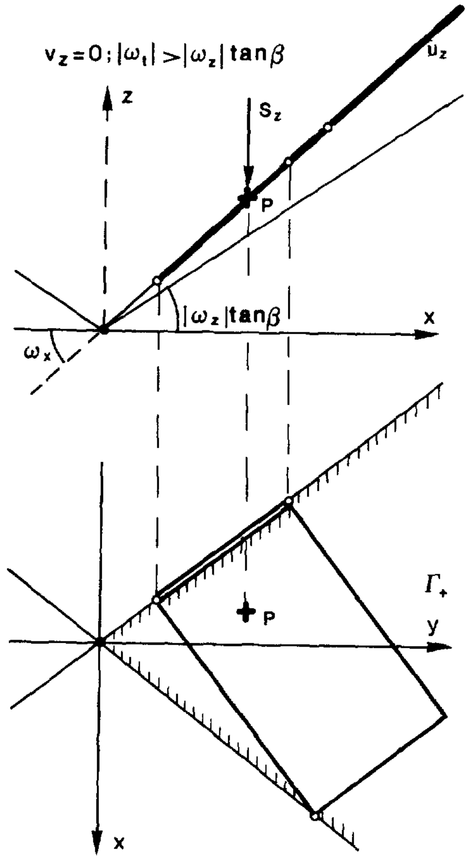

Fig. 6. (b)

Denoting $\kappa = |v_z|/|\omega_z \tan\beta|$; $\delta = |\omega_x|/(|\omega_z|\tan\beta)$, we find the cone to depend only on these two parameters; hence

$$a = \frac{\kappa}{|1-\delta^2|}; \quad b = \frac{\kappa}{|1-\delta^2|^{1/2}}; \quad c = \frac{\kappa\delta}{|1-\delta^2|}; \quad \varepsilon = \delta. \tag{60}$$

This means that the same cone may correspond to different angles β and different deformations $w_{\mu 0} = \{\mathbf{v}_{\mu 0}, \boldsymbol{\omega}_{\mu 0}\}^{\mathrm{T}}$. The cone determines the focal radii r of contact and by the orthogonality $\mathbf{r} \perp \tau(r)$, also the direction of the tangential traction $\tau(r)$, where in the case of contact sliding $|\tau(r)| = |\sigma(r)|\tan\varphi$. Equation (60) implies that the interface forces S depend only on $\varphi = \rho + \beta$ if $\beta > 0$.

The resultant forces $\{S^c\}$, acting on $\Gamma_{0\mu}$, $\{S_z, Q_x, Q_y, M_x, M_y, M_z\}_c^{\mathrm{T}}$ form a cone with deformations, according to (53). The work equation (46), $\langle p_{0\mu}, \gamma_{\mu 0}\rangle_{\partial H0\mu} = \langle S, w\rangle_{\partial G0\mu}$, gives

$$\begin{aligned} \langle S, w\rangle_{\partial G0\mu} &= -|S_z|v_z + \mathbf{M}_t \cdot \boldsymbol{\omega}_t + \mathbf{Q}\cdot\mathbf{v}_t + M_z\omega_z = \int|\sigma||\gamma_t|(\tan\varphi - \tan\beta)\,\mathrm{d}\Gamma \\ &= |\omega_z|\int|\tau|\mathrm{r}\,\mathrm{d}\Gamma(1-\tan\beta/\tan\varphi) \end{aligned} \tag{61}$$

because $|\tau| = |\sigma|\tan\varphi$; $\gamma_n = |\gamma_t|\tan\beta$, and $|\gamma_t| = |\omega_z|r$. The torque with respect to the focus F is $M_{zF} = \pm\int|\tau|r\,\mathrm{d}\Gamma$ because $\tau \perp \mathbf{s}^c - \mathbf{s}_F$, where contact sliding occurs.

If the point of action of S_z, according to Proposition 4(ii), is (x_p, y_p) in the principal system x, y of (58) then, with x in direction ω_t we get

$$\int|\sigma|y\,\mathrm{d}\Gamma = |S_z|y_p; \quad r = a + \varepsilon y; \quad r_p = a + \varepsilon y_p. \tag{62}$$

Hence

$$| M_{zF} | = | S_z | | \omega_z | (a+\varepsilon y_p) \tan \varphi. \tag{63}$$

$$\langle S, w \rangle = | \omega_z | \int | \sigma | r \, d\Gamma (\tan \varphi - \tan \beta) = | S_z | | \omega_z | (a+\varepsilon y_p)(\tan \varphi - \tan \beta). \tag{64}$$

Proposition 12. The limit state that corresponds to a mechanism $w^c \in \partial\Lambda$ depends on the point of action $\mathbf{s}_p$ as follows: (i) If the torque $M_z(s_p) = 0$ where $\mathbf{s}_p$ is within the interior of the convex hull of $\Gamma_{0\mu}$, a limit state $\{S_z^c, Q_x^c, Q_y^c\}^T$ is possible. The corresponding cone $K(\varphi)$ is circular, $(|Q| = |S_z| \tan \varphi)$, being identical to the local cone $\Phi(\varphi)$. If $\mathbf{s}_p$ is on the contour of the convex hull of $\Gamma_{0\mu}$, the rotation axis $\boldsymbol{\omega}_t$ is a tangent of conv $\Gamma_{0\mu}$ through $\mathbf{s}_p$. (ii) If $M_z(s_p) \neq 0$, a limit state is possible only in case $\mathbf{s}_p$ is within the convex hull spanned by the contact points $\mathbf{s}^c$.

The result deduced for a single interface can be generalized to the total assemblage with plane interfaces. Applying the work equation for $\{P'', S'', p''\}^T$ in admissible equilibrium to a mechanism $\{U', w', \gamma'\}^T$ we get with $r' = a' + \varepsilon' y'$ according to (63), (25) and using ordered indexes j for $\mu\nu$ with $S_j = S_{\mu\nu}$ but $\omega_j = \omega_{\nu\mu}$

$$\langle P'', U' \rangle = \sum_j^m | S''_{jz} | \, |\omega'_z| (a' + \varepsilon' y''_p)(\tan \varphi'' - \tan \beta') \leqslant \Sigma | S''_{jz} | (a' + \varepsilon y'_p) | \omega'_{jz} | (\tan \varphi - \tan \beta). \tag{65}$$

The inequality follows from $\tan \varphi'' \leqslant \tan \varphi$ and the presence of dynamic contact points $s^{c'}$ inside Γ'_+, where consequently $r''_c \leqslant a' + \varepsilon' y'_c$. $\{U', w', \gamma'\}^T$ is a mechanism with corresponding principal axes x', y' of the conic and where S''_z and its point of action $\{x''_{jp}, y''_{jp}\}^T$ is defined by

$$\int_{\Gamma_j} |\sigma''| d\Gamma = |S''_{jz}| ; \quad \int_{\Gamma_j} |\sigma''| y' d\Gamma = |S''_{jz}| y''_{jp} ; \quad \int \sigma'' x' d\Gamma = |S''_{jz}| x''_{jp} \tag{66}$$

with $\forall \ (x', y') \in \partial\Gamma'_+$. For corresponding P, U, there holds, according to (65),

$$\langle P, U \rangle_z = \langle S, w \rangle_{\partial G} = \sum_j^m | S_{zj} | (a+\varepsilon y_p) | \omega_{zj} | (\tan \varphi - \tan \beta). \tag{65'}$$

Expressing the load by $P' = P^1 + \lambda' k$ where $P^1 \in E^0(\rho, \beta)$ and $k \notin E(\rho, \beta)$ are given vectors, the intensity λ can, according to (65′), be expressed either by an AK $\{\gamma', \omega', U'\}^T$, where

$$\lambda' = \left(\sum_j^m | S'_{zj} | (a' + \varepsilon' y'_p) | \omega'_{zj} | (\tan \varphi - \tan \beta) + | \langle P^1, U' \rangle | \right) \Big/ \langle k, U' \rangle, \tag{67}$$

or by an admissible equilibrium state $\{P'', S'', p''\}^T$ corresponding to the intensity λ'' with $P'' = P^1 + \lambda'' k$.

Proposition 13. If ρ, β constant with $\beta > 0$ and on each of the interfaces Γ_j one of the following conditions is satisfied: (i) $\rho = 0$; $\varphi = \beta$; (ii) $\omega'_{jz}, v'_{jt} = 0$; no sliding, only hinging; (iii) a', $\varepsilon' = 0$; only one contact point with twisting; (iv) $\varepsilon = 0$; $S''_{jz} = S^c_{jz}$ prescribed; (v) a', $\varepsilon' > 0$; $S''_{jz} = S^c_{jz}$ prescribed and s_{jp} prescribed; then the cone of stability is uniquely determined and the angle of stability θ_k attains extremum values according to Proposition 6, where the angle of friction β is replaced by $\varphi = \rho + \beta$.

Proof. (a″) The upper bound proposition follows from work equations applying any AK $\{U', w', \gamma'\}^T$ to load $P' = P^1 + \lambda' k$ and the actual limit load $P^c = P^1 + \lambda^c k$, which corresponds to equilibrium according to eqns (67) and (65)

$$\langle P^1, U'\rangle + \lambda'\langle k, U'\rangle = \Sigma |S_z'||\omega_z'|(a'+\varepsilon' y_p')(\tan\varphi - \tan\beta)$$

$$\langle P^1, U'\rangle + \lambda^c\langle k, U'\rangle \leqslant \Sigma |S_z^c||\omega_z'|(a'+\varepsilon' y'^c_p)(\tan\varphi - \tan\beta).$$

Subtraction gives

$$(\lambda'-\lambda^c)\langle k, U'\rangle \geqslant \Sigma |\omega_z'|((|S_z'|-|S_z^{c\prime}|)a'+\varepsilon'(|S_z'|y_p'-|S_z^{c\prime}|y_p'^c))(\tan\varphi-\tan\beta);$$

$$\forall\{w', U'\}^{\mathrm{T}} \subset \mathrm{AK}. \tag{67'}$$

The left side of the inequality is non-negative if the right side is zero. This zero condition is satisfied, in cases (i) and (ii) obviously in case (iii) by choosing $\{\mathbf{v}_{\nu\mu}', \boldsymbol{\omega}_{\nu\mu}'\}^{\mathrm{T}}$ so that the locus g contracts into a single point, and in case (iv) by choosing $\{\mathbf{v}_{\nu\mu}, \boldsymbol{\omega}_{\nu\mu}\}^{\mathrm{T}}$ so that g' is a circle, and obviously in case (v). Hence the left side of eqn (67′) is non-negative, from which follows

$$\lambda_+' \geqslant \lambda_+^c; \quad \lambda_-' \leqslant \lambda_-^c. \tag{67''}$$

(b″) The lower bound part is obtained from work equations applying the actual $\{U^c, w^c, \gamma^c\}^{\mathrm{T}} \subset \mathrm{AK}$ to the corresponding state $\{P^c = P^1 + \lambda^c k, S^c, p^c\}^{\mathrm{T}}$ and an AE $\{P'' = P^1 + \lambda'' k, S'', p''\}^{\mathrm{T}}$ according to eqns (65′) and (65)

$$\langle P^1, U^c\rangle + \lambda^c\langle k, U^c\rangle = \Sigma |S_z||\omega_z^c|(a+\varepsilon y_p^c)(\tan\varphi - \tan\beta)$$

$$\langle P^1, U^c\rangle + \lambda''\langle k, U^c\rangle \leqslant \Sigma |S_z''||\omega_z^c|(a+\varepsilon y_p'')(\tan\varphi - \tan\beta).$$

Substraction gives

$$(\lambda^c-\lambda'')\langle k, U^c\rangle \geqslant \Sigma |\omega_z|((|S_z|-|S_z''|)a+\varepsilon(S^c y_p^c - S'' y_p''))(\tan\varphi-\tan\beta). \tag{68}$$

Applying now the procedure of (a″) to the fixed $\{v^c, \omega^c\}^{\mathrm{T}}$ at each interface, we arrive at the same zero condition and finally the lower bound conditions:

$$\lambda_+^c \geqslant \lambda_+''; \quad \lambda_-^c \geqslant \lambda_-''. \qquad \square \tag{68'}$$

An assemblage is considered statically determinate if all interface resultant forces $S_{\mu\nu}$ are determined by equilibrium conditions.

Corollary 13. If the friction is mixed ($\rho,\beta > 0$) and the assemblage is statically determinate, Proposition 4 holds and the cone of stability is uniquely determined.

Proposition 13 is applied to two simple cases on $\Gamma_{0\mu}$:

Example 1. A line section l on $\Gamma_{0\mu}$ with two-point contact at $x_1 = -l/2$, $y_1 = 0$; $x_2 = +l/2$, $y_2 = 0$, loaded by given forces at the origin S_z, $Q_x \neq 0$; $Q_y = 0$ and a torque M_z [Fig. 6(a)]. In order to determine the unknown M_z, we use the maximum principle of Proposition 13:

$$\mathrm{Max}\,(|M_z| = 1|p_y|)$$

with constraints $|S_z| = 2\,|p_z|$; $2\,p_x = Q_x$; $p_x^2 + p_y^2 = p_z^2 \tan^2\varphi$. This gives curve C_s of cone $K(\rho,\beta)$

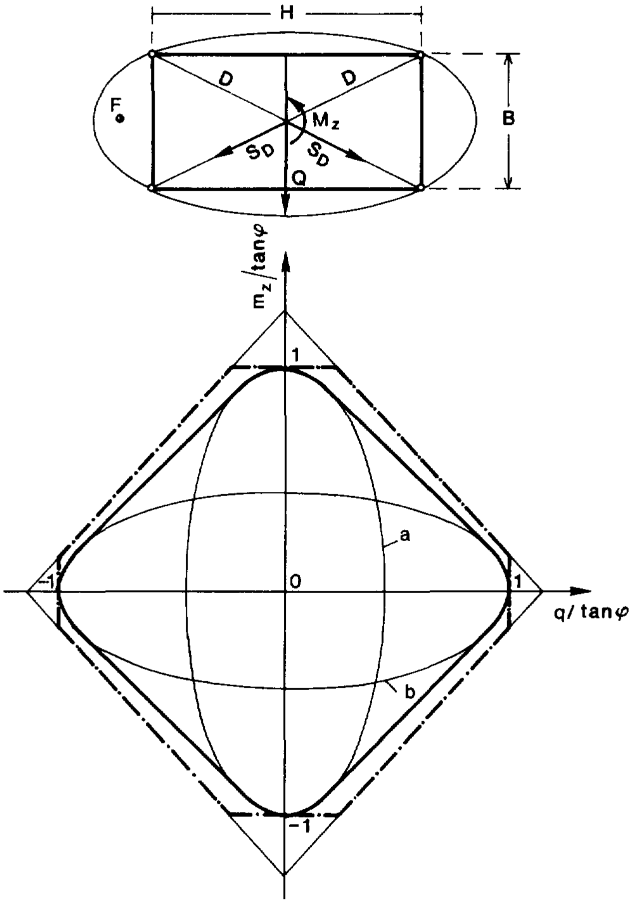

Fig. 7. Base section of cone $K(\rho, \beta)$ of rectangular interface $\Gamma_{0\mu}$ loaded by S_z, Q_x, M_z: $m_z = 2M_z/|DS_z|$; $q = Q_x/|S_z|$; lower bound solution ——; upper bound solution —·—.

$$C_s = \left(\frac{2M_z}{l|S_z|}\right)^2 + \left(\frac{Q_x}{|S_z|}\right)^2 = \tan^2\varphi. \tag{69}$$

The normality rule $\omega_z = \lambda\ (\partial C/\partial M_z)$; $v_x = \lambda\ (\partial C/\partial Q_x)$, gives the pole distance

$$|y_F| = |v_x/\omega_z| = (l^2/4)|Q_x/M_z|.$$

Example 2. From this exact solution, approximative lower bound solutions can be deduced for a rectangular section $B \times H$ with $\mathbf{s}_p$ at the centroid and Q_x in direction B (Fig. 7).

(a) We apply eqn (69) to each of the two diagonals with $l = D$, and forces $p_z = S_z/4$ and normal shear forces $p_t = |p_z|\ \tan\varphi$ at their ends and longitudinal forces S_D with resultant Q_x. This gives (Fig. 7) with $|Q_x| = 2|S_D|B/D$:

$$\left(\frac{2M_z}{DS_z}\right)^2 + \left(\frac{D}{B}\frac{Q_x}{S_z}\right)^2 = \tan^2\varphi. \tag{69a}$$

(b) Another solution is obtained from eqn (69) by decomposing $\{S_z, Q_x, M_z\}^T$ into two equal systems (1) and (2) acting on the two edges $B = 1$ of the rectangle. With $S_{z1} = S_{z2} = S_z/2$; $Q_{x1} = Q_{x2} = Q_x/2$; $M_{z1} = M_{z2} = M_z/2$, eqn (69) applies also to the rectangle

$$\left(\frac{2M_z}{BS_z}\right)^2+\left(\frac{Q_x}{S_z}\right)^2=\tan^2\varphi. \tag{69b}$$

Denoting $m_z = 2M_z/(D|S_z|)$; $q = Q_x/|S_z|$ we thus obtain from eqns (69a) and (69b) two lower bound cones $K''(\varphi)$ represented by two equal ellipses whose main axes are orthogonal

$$\text{(a)}\quad m_z^2+(D/B)^2q^2 \leqslant \tan^2\varphi;\quad \text{(b)}\quad (D/B)^2m_z^2+q^2 \leqslant \tan^2\varphi. \tag{70}$$

Because both solutions are based on AE, and $K(\rho,\beta)$ is convex, the convex hull of (a) and (b) is also a lower bound solution (Fig. 7).

We get an *upper bound solution* from the work equation (64) applied to the elliptic contact at the corner points with $v_{xp} = -\omega_z c$

$$-Q_xc+M_z = (|S_z|/2)(r_1+r_2)\tan\varphi\,\mathrm{sign}\,M_z. \tag{71}$$

In the principal system x,y, the corners satisfy: $(H/2a)^2+(B/2b)^2 = 1$ and $r_1^2 = (c-H/2)^2+(B/2)^2$; $r_2^2 = (c+H/2)^2.+(B/2)^2$. Choosing $c = 0$ and $c \to \infty$, we get, according to Propositions 11 and 12,

$$|M_z| = D/2|S_z|\tan\varphi;\quad Q_x = |S_z|\tan\varphi. \tag{72}$$

Choosing $a = H/\sqrt{2}$; $b = B/\sqrt{2}$, which corresponds to the circumscribed ellipse with the smallest area πab, we get the unknown focal distance $c = |(H^2-B^2)/2|^{1/2}$. This gives

$$|M_z|+((H^2-B^2)/2)^{1/2}|Q_x| = |S_z|(H/\sqrt{2})\tan\varphi. \tag{72'}$$

Expressed by m_z and q_x, eqns (72) and (72′) are

$$m_z = \pm\tan\varphi;\quad q = \pm\tan\varphi;\quad |m_z|+\sqrt{2}((H^2-B^2)/D^2)^{1/2}|q| = \sqrt{2}(H/D)\tan\varphi. \tag{72''}$$

The first two equations represent a pair of lines in the direction of the axes m_z and q, respectively, whereas the third represents two pairs of inclined straight lines. The admissible upper bound solution is bounded by these eight lines (Fig. 7).

10. PURELY DISSIPATIVE FRICTION

The question arises of what happens if $\beta = 0$, and we have purely dissipative friction $\varphi = \rho$. Equation (69) for the segment of line l of Example 1 is an exact solution which is valid for $\beta > 0$. Apparently the cone (69) represents the upper bound $K_a(\varphi)$. What is the lower bound, and is it possible as in plane problems to determine a certainly stable cone $K_s(\varphi,0)$?

If $\beta = 0$, the σ-distribution is undetermined. A statically admissible stress field is one with all the stresses p concentrated on the origin. This gives $M_z = 0$, and $K_s\{M_z,Q_d,S_z\}$ is a zero cone. There is no certainly stable load $\{M_z,Q_x,S_z\}^{\mathrm{T}}$ with each M_z, Q_z, $S_z \neq 0$. The grey zone of neutral loads

$$K_n(S_z,Q,M_z) = K_a(S_z,Q,M_z)-K_a(S_z,Q) \tag{74}$$

comprises almost the total cone $K_a(S_z,Q_x,M_z)$ because the three-dimensional cone K_a is reduced to a plane angle $K_a(S_z,Q)$: $|Q| \leqslant |S_z|\tan\varphi$.

This paradoxical discontinuous drop in stability in the neighbourhood of $\beta = 0$ demonstrates the decisive significance of the dilatational effect manifested by the inclination angle β. If the conditions of Proposition 13 are fullfilled, this discontinuity could be expressed by a drop from $K_a(Q_z,Q_x,M_z)$ to $K_a(S_z,Q_x,0)$ with friction φ. If the conditions (ii)–(v) of Proposition 13 are not satisfied, we can conclude from this that the indefinite zone of neutral equilibrium $E_n(\varphi, \beta)$ is much larger in three-dimensional structures than in plane structures. Especially, no lower bound solution may correspond to certainly safe equilibrium.

11. IMPLICATIONS AND CONCLUSIONS

The ordered sequence of the sets from formula (52),

$$E_s(\rho+\beta, 0) \subset E_s(\rho, \beta) \subset E(0, \rho+\beta) \subset E(\pi/2), \qquad (52'')$$

expressing the transition from dissipative friction to geometric friction simultaneously manifests the gradual transition of the stability criteria from internal dissipative energy to external energy of the load potential. This transition is not merely a quantitative one, but rather a transition from a state of indefiniteness associated with dissipative friction to a state where stability is uniquely determined in the simplest way. The corresponding linear theory of stability of assemblages of rigid bodies, where the deformations and internal forces are subjected to polar conical restraints, completely ignores the restrictions imposed on the carrying capacity by the limited strength and elastic or dissipative phenomena. Nevertheless, this "theory of stone structures" forms the rationalized core of the statics of masonry. Because the normality rule is valid at every joint, the cone of stability $E(0, \varphi)$ coincides with cone $E_a(\varphi)$. Since this theory thus provides the largest range of stable loads, it may be considered as an ideal model, and the structural object of design is therefore to ensure the conditions presupposed by the theory. Although this theory of nondissipative friction is the only one physically consistent with the rigidity and infinite strength of its structural units, it conforms with reality only if the displacements are infinitesimal. But this is a sufficient condition for incipient collapse in a rigid body assemblage.

The presence of dissipative friction ρ alters radically the behaviour of the loaded structure. Instead of a unique limit surface $\partial E(0, \varphi)$, the cone $E(\rho, \beta)$ includes a large range of neutral states, which makes the assessment of the stable range very difficult.

One method to estimate the boundaries of the cone $E(\rho, \beta)$ is to find out the range of neutral states corresponding to a given load. Neutral states with contact sliding are possible within an interval of the friction angle $(\varphi_{\min}, \varphi_{\max})$. $\varphi_{\min}$ corresponds to the highest angle of stability $\theta_{\max}$ and $\varphi_{\max}$ to the smallest angle $\theta_{\min}$. The difference $\theta_{\max} - \theta_{\min}$ defines the range of indefiniteness and $\theta_{\min}$ defines the boundary of the *certainly safe cone* $E_s(\rho, \beta)$ of stability. This provides satisfactory results in plane problems (Fig. 4).

In three-dimensional cases combined with twisting, this method is not applicable, as is seen from the examples in Section 10. Recently Livesley called attention in his interesting paper (1992) to the same great ambiguity combined with torsion. He has developed a computer scheme based on the equilibrium and the friction condition, which he solves as a lower bound problem by linear programming. Because the nodes used are placed at the corner points of the rectangular cross-sections, they correspond to the contact points, according to Figs. 5 and 6, of nondissipative friction. The question arises: is Livesley's solution a true lower bound solution or has there been excluded neutral states, corresponding to still narrower ranges of stability $E_s(\varphi)$? Are the conditions of Proposition 13 really satisfied, which, owing to the ambiguous formulation of the proposition, is not quite clear. Livesley's solution corresponds, in any case, completely to that of nondissipative friction with $\varphi = \beta$. The paradoxical drops of stability in the neighbourhood of the zero value of the inclination angle β is a riddle that should be investigated experimentally by the twisting of disks in compression. Friction tests have, to our knowledge, been carried out mostly by experiments with pure translation.

In traditional stone structures, the sensitivity to the distribution of the compressive stresses is taken into consideration in many ways. The use of families of linear arches is based on plane thrust lines in the analysis of vaults. Often two families with mutually orthogonal pressure lines are used. If the two families are not interdependent, which means

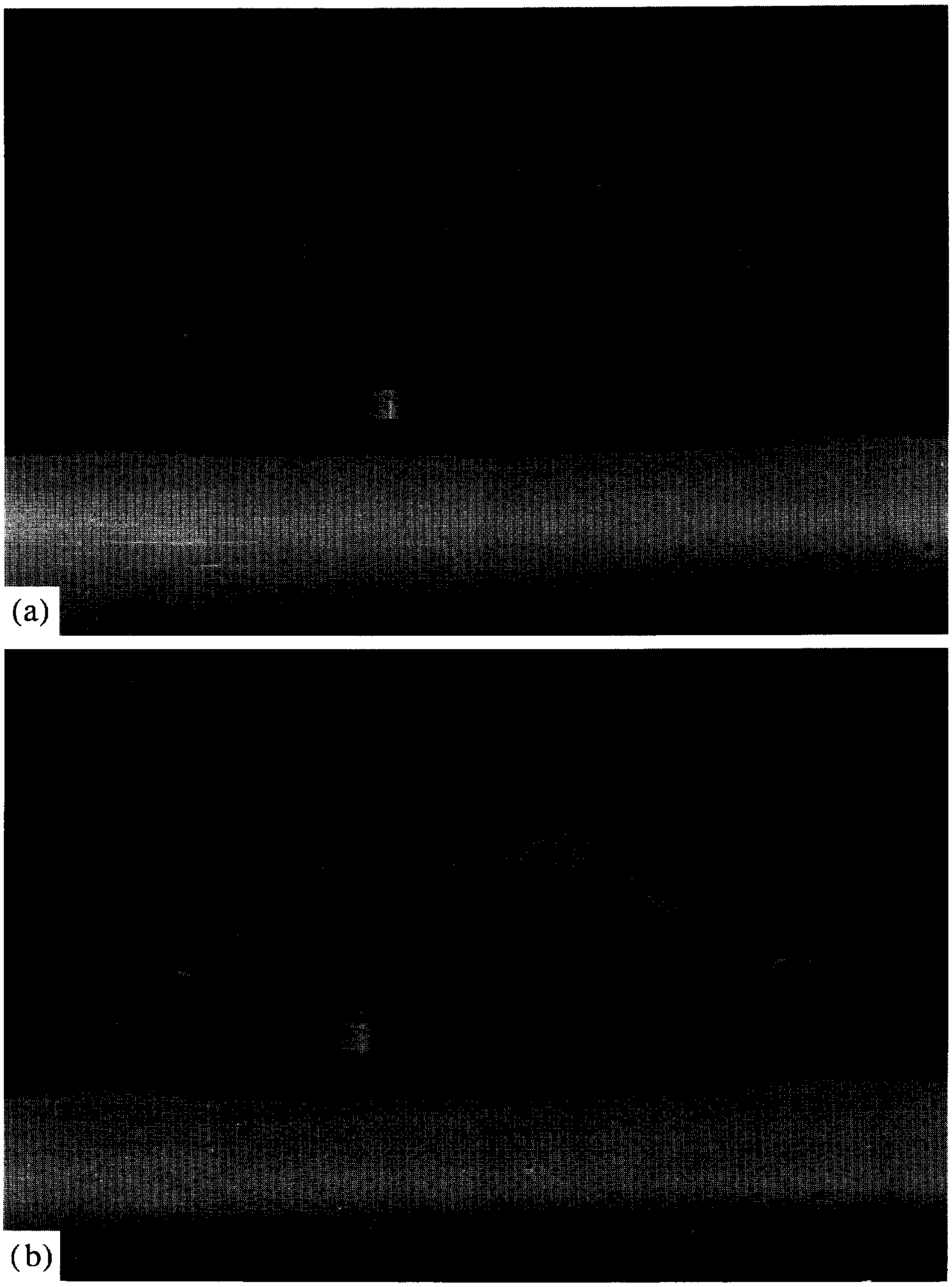

Fig. 8. (a). Thick-walled arches with plane interfaces are susceptible to sliding collapse; (b) by making the arch thinner, the sliding is replaced by hinging, which eliminates the indefiniteness connected with sliding.

that each family carries a portion of the load independently, the pressure lines of the families may lie in different surfaces. The two families are interdependent if there is any kind of force transfer between them as between lateral and meridional pressure lines in domes. In order to avoid the development of wrenches in the contact faces, the linear arches of the two families must lie within the same membrane surface. This occurs in traditional masonry, where the orientation of brick-courses and mortar beds follows certain thrust-line patterns according to the requirement of certain stability. On the other hand, torsional effects on smooth interfaces may contribute to slip failure of thick walled stone arches and vaults. This trend is clearly perceptible in the developments and the structural form of traditional masonry. Particularly, the lead in column joints ensures a uniform stress distribution and therefore a nonzero torsional capacity enlarging the stable range above $E_{min}(\varphi, 0)$.

The indefiniteness of the stable range can thus be eliminated by constructional means. In plane structures, the linear arches provide very efficient stability criteria. In the case of nondissipative friction, if twisting is avoided, Corollary 6 to Proposition 6 provides the simple rule:

Proposition 14. An arch or vault is stable at a given load if there exists at least one safe linear arch or one family of safe linear arches, respectively, which lies everywhere within the masonry and within the friction cone at every joint.

This rule, corresponding to geometric friction, ought to be modified if the friction is dissipative.

Proposition 15. An arch or vault is certainly stable at a given load if all the lines of thrust lying everywhere within the masonry have tangent vectors that lie within the friction cone at every joint.

In thick-walled arches, the susceptibility to sliding has been a difficult matter to deal with because the normality rule ceases to be valid in this case. The conditions $E(\rho, 0) = E_a(\rho)$ and that of normality may be restored or at least approached either by keying the joints, with sliding according to mixed friction, or by making the arches thinner, whereby sliding is completely prevented and the structure is transformed into a hinge mechanism (Fig. 8). Keying corresponds to a shift $E(\rho, 0) \Rightarrow E(\rho, \beta)$, whereas the reduction of the thickness corresponds to a shift $E(\rho, 0) \Rightarrow E_a(\pi/2)$. On this latter shift is based the use of families of linear arches for lower bound solutions of vaults, developed by Heyman (1966).

The conical shape and convexity of the stable load range are manifested in many particular features of stone structures. The convexity indicates that if safe or certainly safe lines of thrust with ordinates y_1, y_2 correspond to two stable loads P_1 and P_2, the load $P = \lambda P_1 + \lambda_2 P_2$ with $\lambda_1, \lambda_2 > 0$ is also stable because it corresponds to a safe line of thrust $y(x)$, situated everywhere in the interval $[y_1, y_2]$. This conic convexity expresses a certain law of superposition that completely contradicts the principles of plasticity (Parland, 1979).

In stone structures, high concentrations of gravity masses at the bends of vaults and arches are needed to compel the thrust lines to remain within the masonry. In this case, the potential energy of the gravity is substituted for the strain energy concentrations in the corresponding monolithic structure. This leads to a completely different weight distribution with a concentration of gravity masses at great heights, contrary to the principles of strength of material. This appears also in the stone columns of Gothic churches. The anomalous weight distribution is due to the unboundedness and the convex–conical shape of the range of stable loads. The erection of large-span arches and vaults in the Middle Ages, without preceding design calculations, becomes understandable because of the simplicity of the stability conditions. Owing to the polarity of the cones E and Ξ and the coincidence of E and E_a, nondissipative friction leads to extremely simple design rules which, together with the arbitrariness of the stable equilibrium, made possible the great variety of the structural forms of traditional masonry. The main importance of the introduction of a dilatational nondissipative component β into the friction law is therefore not its physical credibility but the uniqueness and upperbound characteristics it provides for the evaluation and the improvement of the stability of stone structures.

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